2022 Honors Algebra Exam

Instructions: This exam has three parts: I, II, and III, each with three problems. Try to fully solve **two problems from each part**. Then once you are not making progress at these six chosen problems, give another pass at the exam and try to solve the remaining three problems. I am interested in seeing how you approach the various problems, so please turn in your solution to a problem even if you can only make partial progress on it. For the problems with multiple parts, if you cannot complete a part, still attempt to complete latter parts assuming the earlier ones.

Part I consists of conceptual problems. They should require knowing definitions of terms and broad understanding of the topic in question. Part II consists of multipart problems in the areas of groups, rings, and Galois theory and part III consists of single part problems in the areas of groups, rings, and Galois theory.

Part I

(I. Prob. **1**) In Algebra, there are several instances where we "quotient" an object by another object. For example, given a group G and a normal subgroup H, we create the quotient group G/H; given a ring R and an ideal I, we create the quotient ring R/I; given a module M and the submodule N, we create the quotient module M/N. Conceptually, provide an explanation of this "quotienting" process (e.g. what are the elements of the quotient space, why is the quotient space also a group/ring/module, and how should we think of the quotient space). Include at least one example that illustrates your explanation.

(I. Prob. **2**) For this problem, you can assume we know the definition of a ring with unity.

(a) Define each of the following terms: commutative ring, Euclidean domain, integral domain, unique factorization domain, principal ideal domain, and field.

(b) Write a collection of statements of the form: every A is also a B, every B is a C, every C is a D, every D is an E, and every E is an F, where each letter is one of the terms in part (a).

(c) Give an example of why each of the five statements above are true only in the direction stated. That is, provide an example of a ring that is a B but not an A, a ring that is a C but not a B, and so on.

(I. Prob. **3**) Let G be a group.

(a) Define a representation of G on a vector space V over a field K.

(b) Describe the character table of a group G. You have some freedom about what to write about based on what you think is important about character tables. As a suggestion, here are some questions you could discuss: What information is included in the table? What are the rows and columns indexed by? Why do we only include conjugacy classes instead of all elements of G? Why do we only include irreducible representation? What is the importance of character tables in the study of groups?

(c) Present the character table of S_3 with its three irreducible representations.

Part II

(II. Prob. **4**) Let G be a group and $Z(G)$ be its center.

(a) Show that if $H \leqslant Z(G)$ and G/H is cyclic, then G is abelian.

(b) Let $Aut(G)$ be the group of automorphisms of G and $Inn(G) = \{\phi_g \mid g \in G\}$ where $\phi_g : G \to G$ such that $\phi_g(x) = gxg^{-1}$. Prove that $Inn(G) \cong G/Z(G)$. [Hint: First Isomorphism Theorem.]

(c) Prove that if $Aut(G)$ is cyclic then G is abelian. [Hint: Use (a) and (b)].

(II. Prob. **5**) Let $f : R \to S$ be a homomorphism of commutative rings.

(a) Show that if P is a prime ideal of S, then its preimage $f^{-1}(P)$ is a prime ideal of R.

(b) Show that if f is surjective and M is a maximal ideal of S, then its preimage $f^{-1}(M)$ is a maximal ideal of R.

(c) Give an example of a non-surjective $f : R \to S$ and a maximal ideal M of S such that $f^{-1}(M)$ is not a maximal ideal of R.

(II. Prob. **6**) Let $\alpha = \sqrt{2 + \sqrt{2}}$.

(a) Find the minimal polynomial p of α. [Eisenstein criterion might be useful to show irreducibility.]

(b) Show that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a Galois extension (also known as a finite normal extension).

(c) Find the Galois group of the Galois extension $\mathbb{Q}(\alpha)/\mathbb{Q}$.

Part III

(III. Prob. **7**) Classify all groups of order 75.

(III. Prob. **8**) An integral domain D is said to be Artinian if for any descending chain of ideals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

of D, there is an integer n such that $I_i = I_n$ for all $i \geq n$. Prove that an integral domain D is Artinian if and only if it is a field.

(III. Prob. **9**) Let K be a field of characteristic $p > 0$ and E/K be a field extension. Let α be a root in E of an irreducible polynomial $f(x) = x^p - x - a$ for some $a \in K$ (so $f \in K[x]$). Show that f is separable and that the Galois group of the splitting field of f over K is cyclic of order p. [You can assume that the splitting field of a separable polynomial is a Galois extension (also known as a finite normal extension).]