SWARTHMORE COLLEGE
DEPARTMENT OF MATHEMATICS AND STATISTICS
2020 ALGEBRA HONORS EXAMINATION

Instructions: This exam contains nine problems. Try to solve **six** problems as completely as possible. The exam is divided into three sections; please attempt to solve at least one, ideally more than one, problem from each section. Once you are satisfied with your responses to six problems, make a second pass through the exam and complete as many parts of the remaining problems as possible. For problems with parts, you are allowed to assume the truth of earlier parts when working on a later part. I am interested in your thoughts on a problem and attempts at special cases even if you do not completely solve the problem. Please justify your reasoning as fully as possible.

Section I.

1. The set $\mathrm{Aut}(G)$ consisting of all automorphisms of a group G (i.e. isomorphisms from G to G) is a group under composition of functions. If $\alpha \in \mathrm{Aut}(G)$ and $x \in G$ let x^α denote the image of x under α. (If α, $\beta \in \mathrm{Aut}(G)$ then $x^{\alpha\beta} = (x^\alpha)^\beta$ so apply α first, β second.)

(a) Let H be a subgroup of G such that $h^\alpha \in H$ whenever $\alpha \in \mathrm{Aut}(G)$ and $h \in H$. Let $\mathrm{Aut}^H(G)$ be the set consisting of all $\alpha \in \mathrm{Aut}(G)$ such that for every $x \in G$ we have $x^\alpha = xh$ for some element $h \in H$ that depends on x. Prove that $\mathrm{Aut}^H(G) \triangleleft \mathrm{Aut}(G)$.

(b) For each $g \in G$ the automorphism $\varphi_g \in \mathrm{Aut}(G)$ is defined by $x^{\varphi_g} = g^{-1}xg$ for all $x \in G$. We call φ_g the inner automorphism of G induced by g. It is known (do not prove this) that the set $\mathrm{Inn}(G) = \{\, \varphi_g \mid g \in G \,\}$ is a normal subgroup of $\mathrm{Aut}(G)$. Prove that $\mathrm{Aut}^{\mathbf{Z}(G)}(G)$ is equal to the set consisting of all those $\alpha \in \mathrm{Aut}(G)$ that commute with every inner automorphism of G. Here $\mathbf{Z}(G)$ denotes the center of G.

(c) For any subset S of G and any $\alpha \in \mathrm{Aut}(G)$ we define the set $S^\alpha = \{\, x^\alpha \mid x \in S \,\}$. Let $\mathrm{Cl}(G)$ be the set whose members are the distinct conjugacy classes of elements of G. Prove that $K^\alpha \in \mathrm{Cl}(G)$ whenever $K \in \mathrm{Cl}(G)$ and $\alpha \in \mathrm{Aut}(G)$. Then argue that each $\alpha \in \mathrm{Aut}(G)$ permutes the members of $\mathrm{Cl}(G)$, so the group $\mathrm{Aut}(G)$ acts on the set $\mathrm{Cl}(G)$.

(d) For $x \in G$ let x^G denote the conjugacy class of elements of G that contains x. An automorphism $\alpha \in \mathrm{Aut}(G)$ is said to be class-preserving if $x^\alpha \in x^G$ for all $x \in G$. Let $\mathrm{Aut}_c(G)$ be the set of all class-preserving automorphisms of G. Prove $\mathrm{Aut}_c(G) \triangleleft \mathrm{Aut}(G)$.

2. Prove that there does not exist a simple group of order 336. (Note $336 = 2^4 \cdot 3 \cdot 7$.) Hint: G acts via conjugation on the set $\mathrm{Syl}_p(G)$ for any prime p dividing $|G|$. This yields a map from G to a symmetric group S_n for a certain n.

3. Let $\Omega = \{1, \ldots, n\}$ with $n > 1$. We write i^α to denote the image of any $i \in \Omega$ under any permutation α in the symmetric group S_n. (If α, $\beta \in S_n$ then $i^{\alpha\beta} = (i^\alpha)^\beta$ so apply α first, β second.) Let I be the set of all ordered pairs of integers (a, b) such that $0 \leq b < n$ and $1 \leq a \leq n$ while a and n are relatively prime. For each $(a, b) \in I$ the "affine map" $\pi_{a,b} : \Omega \to \Omega$ is defined by $i \mapsto j$ where $ai + b \equiv j \pmod{n}$. Let $A = \{\, \pi_{a,b} \mid (a, b) \in I \,\}$.

(a) Prove that the map $I \to A$ given by $(a, b) \mapsto \pi_{a,b}$ is bijective. What is $|A|$?

(b) Prove that A is a subset of the symmetric group S_n.

(c) Let σ denote the n-cycle permutation $(\, 1 \;\; 2 \;\; \cdots \;\; n \,)$ in S_n. Prove that $\alpha^{-1}\sigma\alpha$ is the n-cycle permutation $(\, 1^\alpha \;\; 2^\alpha \;\; \cdots \;\; n^\alpha \,)$ for each $\alpha \in S_n$.

(d) Let N denote the normalizer in S_n of the cyclic group $\langle\sigma\rangle$. Prove that $A = N$. Hint: Begin by arguing directly that $|A| = |N|$.

Section II.

4. In this problem R always denotes a commutative ring with unity. Recall that if A and B are ideals of R then each of the sets $A \cap B$, $A + B$, AB is an ideal of R. Recall that AB is the set of all finite sums of the form $a_1 b_1 + \cdots + a_n b_n$ with $a_i \in A$ and $b_i \in B$. Recall that $A + B$ is the set of all sums of the form $a + b$ with $a \in A$ and $b \in B$.

(a) Give an example of a ring R and a pair of nonzero proper ideals A, B of R such that $A \cap B \neq AB$.

(b) Let A and B be ideals of R such that $A + B = R$. Prove that $A \cap B = AB$.

(c) Let M_1, M_2, $\ldots$, M_r be distinct maximal ideals of R. Prove that $M_1 M_2 \cdots M_r = M_1 \cap M_2 \cap \cdots \cap M_r$.

5. Let T be a ring with unity. Let $S = M_n(T)$ be the ring of all $n \times n$ matrices over T.

(a) If J is an ideal of S, prove that $J = M_n(I)$ for some ideal I of T. Hint: Use matrices of the form e_{pq} whose entry in row p, column q is 1 while all other entries are 0.

(b) A ring element x is said to be nilpotent if $x^n = 0$ for some positive integer n. The trace of a matrix $s \in S$ is defined as the sum of all its diagonal entries and is denoted by $\mathrm{tr}(s)$. If T is a field and $s \in S$ is nilpotent, prove that $\mathrm{tr}(s) = 0$.

(c) Suppose T is a field. Suppose R is a ring such that $\theta : R \to S$ is a surjective homomorphism. Let I be an ideal of R with the property that every element of I is a sum of nilpotent elements of R. Prove that $\theta(x)$ is the zero matrix for every $x \in I$.

6. Let χ be a character of degree n of a finite group G. Recall that there may be many distinct representations of G whose character is χ. Choose one such representation and denote it by X. Let $\lambda : G \to \mathbb{C}$ be the homomorphism (linear character) defined by $\lambda(g) = \det(X(g))$, the determinant of the matrix $X(g)$.

(a) We say that λ is the determinant of χ and write $\lambda = \det(\chi)$. Show that this is well-defined. In other words, show that λ is independent of the choice of X.

(b) For the restriction of χ to an abelian subgroup A of G we write $\chi_A = \mu_1 + \cdots + \mu_n$ where each μ_i is a linear character of A. Prove that λ_A is equal to the product $\mu_1\mu_2\cdots\mu_n$.

(c) Let $z \in \mathbf{Z}(G)$ be an element of order m. $\mathbf{Z}(G)$ denotes the center of G. Suppose χ is irreducible and that the homomorphism $X : G \to \mathrm{GL}_n(\mathbb{C})$ is injective. Prove that $X(z) = \varepsilon I$ where I is the identity matrix and ε is a primitive complex mth root of unity.

(d) Prove that m divides n if we further assume $z \in G'$ in the situation of part (c). G' is the subgroup of G generated by all the elements of the form $x^{-1}y^{-1}xy$ where $x, y \in G$.

(e) Let $t \in G$ be an element of order 2. Prove that $\chi(t) \in \mathbb{Z}$ and $\chi(t) \equiv n \pmod 2$.

(f) Prove that $\chi(t) \equiv n \pmod 4$ if we further assume $t \in G'$ in the situation of part (e).

Section III.

7. Let L be a splitting field of the polynomial $x^4 + 8$ over $\mathbb{Q}$.

(a) Show that $x^4 + 8$ is irreducible over $\mathbb{Q}$.

(b) Find $|L : \mathbb{Q}|$.

(c) Describe the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ by stating it is isomorphic to a familiar group.

8. Show that there exists a Galois extension L of $\mathbb{Q}$ such that the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ is noncyclic of order 9. Hint: If ω is a complex primitive 7th root of unity then $\mathbb{Q}(\omega)$ contains a field which is Galois over $\mathbb{Q}$ for which the Galois group has order 3. This doesn't do it but you can find a subfield of a suitable cyclotomic field that does.

9. Let $E = \mathbb{Q}(\omega)$ where $\omega = e^{2\pi i/p} \in \mathbb{C}$ for some odd prime p. Consider the group $U(p) = \{1, 2, \ldots, p-1\}$ under multiplication modulo p. Fix an element $1 \neq k \in U(p)$ and let H be the cyclic subgroup generated by k. Let α be the unique automorphism in the Galois group $\mathrm{Gal}(E/\mathbb{Q})$ that satisfies $\alpha(\omega) = \omega^k$. Note that $L = \{\, c \in E \mid \alpha(c) = c \,\}$ is the fixed field for the subgroup $\langle\alpha\rangle$ of $\mathrm{Gal}(E/\mathbb{Q})$. Thus we have $\mathbb{Q} \subseteq L \subseteq E$.

(a) Show that the cyclic groups $\langle\alpha\rangle$ and H have the same order.

(b) For each subset S of $U(p)$ we define the complex number $\sigma(S)$ by letting

$$\sigma(S) = \sum_{i \in S} \omega^i.$$

Let T be a set of representatives for the cosets of H in $U(p)$. Let $\mathcal{B} = \{\, \sigma(tH) \mid t \in T \,\}$. In the special case $p = 13$ and $k = 3$ write down the members of $\mathcal{B}$ explicity, then verify by direct calculation that $\alpha(b) = b$ for each $b \in \mathcal{B}$, and then conclude that $\mathcal{B} \subseteq L$.

(c) Prove that $\mathcal{B} \subseteq L$ in the general situation where p and k are arbitrary.

(d) For arbitrary p and k, prove that $\mathcal{B}$ is a basis for the vector space L over $\mathbb{Q}$.