Swarthmore College
Department of Mathematics and Statistics
Honors Examinations in Topology 2020

Instructions: Do as many of the following problems as thoroughly as you can in the time you have. Include at least one problem from each of the four parts of the exam.

Part I: Point Set Topology

1) Let X be the set of all integers $n > 1$. For each $n > 1$, let $U_n \subseteq X$ be the subset of all divisors of n. Let τ be the smallest topology on X such that $U_n \in \tau$ for each $n > 1$.
   a) Show that (X, τ) is not locally compact.
   b) Show that (X, τ) is not Hausdorff.
   c) Show that (X, τ) is path connected.

2) Consider the action of $G = \mathbb{Z} \times \mathbb{Z}$ on $\mathbb{R}^2$ defined by $(m, n)(x, y) = (x + m, (-1)^n y)$ for all $(m, n) \in G$ and $(x, y) \in \mathbb{R}^2$. Let $X = \mathbb{R}^2/G$ and let $p : \mathbb{R}^2 \to X$ be the quotient map.
   a) Show that there exists $x \in \mathbb{R}^2$ such that for every neighborhood U of x and every $g_1, g_2 \in G$ the intersection $(g_1 \cdot U) \cap (g_2 \cdot U)$ is non-empty (i.e. prove that the action of G on $\mathbb{R}^2$ is not properly discontinuous).
   b) Show that X is a non-compact topological surface with boundary.
   c) Give an example of a non-compact subset Y of $\mathbb{R}^2$ such that $p(Y)$ is compact.

3) Let ℓ^2 be the metric space of sequences $x = (x_1, x_2, \ldots)$ or real numbers such that $\sum_{i=1}^{\infty} x_i^2$ with distance

$$d(x, y) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$$

   for all $x, y \in \ell^2$.
   a) Show that d is indeed a metric on ℓ^2 and that (ℓ^2, d) is a complete metric space.
   b) Show that the subset

$$X = \left\{ x \in \ell^2 \,\middle|\, \sum_{n=1}^{\infty} n^2 x_n^2 \leq 1 \right\}$$

   is a compact subset of ℓ^2.
   c) Show that 0 is not in the interior of X.

Part II: Homotopy

4) Let $X_1, \ldots, X_n$ be compact oriented connected topological surfaces of genus $g_1, \ldots, g_n$ (respectively). For each $i \in \{1, \ldots, n\}$ let $f_i : \{1, \ldots, m\} \to X_i$ be arbitrary functions and functions. Let Z be the space obtained by quotienting the disjoint union $X_1 \sqcup X_2 \sqcup \cdots \sqcup X_n$ by the identification $f_i(N) \sim f_j(N)$ for all $N \in \{1, \ldots, m\}$ and for all $i, j \in \{1, \ldots, n\}$.
   a) Show that Z is connected and calculate $\pi_1(Z)$.
   b) Classify the spaces Z thus obtained up to homotopy.

c) What happens if the orientability assumption is removed? What other generalizations can you make?

5) Calculate the fundamental group of the following spaces:

a) $X = \{(z, w) \in \mathbb{C}^2 \,|\, (z\bar{z} - 1)(w\bar{w} - 1) = 0\}$

b) $Y = \{(z_1, z_2, w) \in \mathbb{C}^3 \,|\, (z_1\bar{z}_1 + z_2\bar{z}_2 - 1)(w\bar{w} - 1) = 0\}$

c) $Z = \{(z_1, z_2, w_1, w_2) \in \mathbb{C}^4 \,|\, (z_1\bar{z}_1 + z_2\bar{z}_2 - 1)(w_1\bar{w}_1 + w_2\bar{w}_2 - 1) = 0\}$

6) Let $X = \left\{ \left[\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right] \,\middle|\, ac - b^2 \neq 0 \right\}$ viewed as a subspace of the space of all 2×2 matrices (with its standard euclidean topology).

a) Show that X is not connected.

b) Calculate the fundamental group of each connected component.

c) Show that X is homotopic to the space of configurations of two distinct straight lines in the plane.

Part III: Covering Spaces

7) Let $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ be covering maps.

a) Show that $p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2$ defined by $p_1 \times p_2(x_1, x_2) = (p_1(x_1), p_2(x_2))$ for all $(x_1, x_2) \in E_1 \times E_2$ is a covering.

b) Given a covering map $p : E \to B$, we denote by $\mathcal{C}(E, p, B)$ the group of covering transformations i.e. all maps $f : E \to E$ such that $p \circ f = p$. Show that if B_1 and B_2 are connected, then $\mathcal{C}(E_1 \times E_2 \times, p_1 \times p_2, B_1 \times B_2) \cong \mathcal{C}(E_1, p_1, B_1) \times \mathcal{C}(E_2, p_2, B_2)$.

c) What happens if the connectedness assumption is removed?

8) Let $C_p \subseteq \mathbb{R}^2$ be the circle of radius 1 centered at $(p, 0)$. Describe explicitly all the connected regular double covers of $C_{-2} \cup C_0 \cup C_2$.

9) Let $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\}$ and let G be the subgroup of $\mathrm{Aut}(S^3)$ generated by f and g such that $f(z_1, z_2) = (z_2, z_1)$ and $g(z_1, z_2) = (\sqrt{-1}z_1, -\sqrt{-1}z_2)$. Show that S^3/G is compact and connected. Show that the quotient map is a covering. Classify the regular connected coverings of S^3/G.

Part IV: Essays

10) Write a short essay comparing the theory of metric spaces with the theory of topological spaces. Include at least one proof.

11) Write a short essay on the intuitive notion of connectedness and how it is axiomatized in topology. Include at least one proof and one non-trivial example.

12) Write a short essay on the role of the fundamental group in the theory of covering spaces. Include at least one proof.