Real Analysis Honors Exam
Spring, 2020

Note: Integration refers to Riemann integration, either on closed intervals, $[a,b] \subset \mathbb{R}$, or its extension to improper integrals on closed half-lines, $[a,\infty)$.

1. Let $n \in \mathbb{N}$ be odd and $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $a_n \neq 0$, an nth degree polynomial with real coefficients. Prove that $p : \mathbb{R} \to \mathbb{R}$ is surjective (onto).

2. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on an interval $[a,b] \subset \mathbb{R}$. Suppose that $\{f_n\}$ converges uniformly on $[a,b]$ as $n \to \infty$. Prove that $\lim_{n\to\infty} \int_a^b f_n(x)\,dx$ exists.

3. Let $\phi : \mathbb{Z} \to [0,\infty)$, and define a function $d : \mathbb{Z} \times \mathbb{Z} \to [0,\infty)$ by $d(n,m) = \phi(n-m)$, $\forall n,m \in \mathbb{Z}$.

   (i) Find nontrivial condition(s) on ϕ implying that d is a metric on $\mathbb{Z}$.

   (ii) For ϕ satisfying the condition(s) from (i), find additional nontrivial condition(s) implying that $(\mathbb{Z}, d)$ is a complete metric space.

4. The space of continuous functions $C[a,b]$ on a closed, bounded interval $[a,b] \subset \mathbb{R}$ is known to be a complete metric space with respect to

$$\rho(f,g) = \sup_{a \le x \le b} |f(x) - g(x)| .$$

Let $[a,b] \times [a,b] = \{(x,y) \mid x,y \in [a,b]\} \subset \mathbb{R}^2$, and suppose Suppose that $K \in C([a,b] \times [a,b])$ is a continuous, $\mathbb{R}$-valued function. Show that the mapping T defined by $(Tf)(x) = \int_a^b K(x,y) f(y)\, dy$ is well defined, $T : C[a,b] \to C[a,b]$, and, with respect to the metric space topology on $(C[a,b], \rho)$, is a continuous mapping, $T : C[a,b] \to C[a,b]$.

5. A function $H : \mathbb{R}^n \to \mathbb{R}$ is **Lipschitz continuous** if $\exists\, M$ such that $|H(t) - H(s)| \le M|t - s|$, for all $t, s \in \mathbb{R}^n$, $|t - s| \le 1$.

   Let $f_1, f_2 \in C[a,b]$ as above, and define

$$H(t_1, t_2) = \sup_{a \le x \le b} |t_1 f_1(x) + t_2 f_2(x)|.$$

   Prove that $H : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz continuous.

6. Suppose that $\gamma : [a,b] \to \mathbb{R}^3$ be an embedded smooth curve with image Γ, i.e., γ is 1-to-1, smooth (C^∞) and $\dot{\gamma}(t) \neq 0$, $\forall t \in [a,b]$, and $\Gamma = \gamma([a,b])$. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function whose gradient $\nabla f(x)$ is perpendicular to the tangent line of Γ at each point $x \in \Gamma$. Prove that f is constant on Γ.

7. Suppose M is an n-dimensional smooth (i.e., C^∞) manifold and ω is a smooth differential k-form on M.

   (i) Prove that if $k > n$, then $\omega = 0$.

   (ii) Suppose that $n = 2m$ and ω is a 2-form such that $\omega^n := \omega \wedge \omega \wedge \cdots \wedge \omega$ (m times) has the property that $\int_U \omega^n > 0$ for all open sets $U \subseteq M$. Prove that for every $x \in M$, $\omega|_{T_x M}$ is a nondegenerate bilinear form, i.e., for every $x \in M$ and every $v \in T_x M$, there exists a $u \in T_x M$ such that $\omega(u,v) \neq 0$.

(continued)

8. Let Γ be a smooth embedded curve in $\mathbb{R}^2$ which is closed, i.e., such that $\gamma(a) = \gamma(b)$. (See problem 6 for the notation.) Prove that there exists a smooth function $F : \mathbb{R}^2 \to \mathbb{R}$ such that $\{x \in \mathbb{R}^2 \,:\, F(x) = 0\} = \Gamma$.