Swarthmore College
Department of Mathematics and Statistics
Honors Examinations in Geometry 2020

Instructions: Do as many of the following problems as thoroughly as you can in the time you have. Include at least one problem from each of the three parts of the exam.

Part I: Curves

1) A *Bertrand curve* is a parametrized curve $\gamma : [0,1] \to \mathbb{R}^3$ for which there exists a distinct parmetrized curve $\tilde{\gamma} : [0,1] \to \mathbb{R}^3$ such that the straight line through $\gamma(t)$ and $\tilde{\gamma}(t)$ is normal to both γ and $\tilde{\gamma}$. Show that if this is the case then $||\gamma(t) - \tilde{\gamma}(t)||$ is a constant function of t. Show that an arclength parametrized curve γ with nowhere vanishing curvature and torsion is a Bertrand curve if and only if there exist constants $a, b \in \mathbb{R}$ such that $a\kappa + b\tau = 1$.

2) Calculate the curvature and torsion of the curve $\gamma : [0,1] \to \mathbb{R}^3$
$$\gamma(t) = (2t + \cos(t), t - 2\cos(t), \sqrt{5}sin(t)) .$$
Find an isometry of $\mathbb{R}^3$ that maps γ into a cylindrical helix whose tangent makes a constant angle with the third coordinate axis.

3) Show that the subset of $\mathbb{R}^3$ obtained by intersecting the cylinder $x^2 + y^2 = 1$ with the sphere $(x-1)^2 + y^2 + z^2 = 1$ is can be parametrized by a regular curve. Calculate the curvature and the torsion at the point $(0, 1, 0)$.

Part II: Surfaces

4) The third fundamental form $\mathbf{III}$ of a surface in $\mathbb{R}^3$ is defined by $\mathbf{III} = d\mathbf{N} \cdot d\mathbf{N}$, where $\mathbf{N}$ is the unit normal. Show that $K\mathbf{I} - 2H\mathbf{II} + \mathbf{III} = 0$, where K is the Gaussian curvature, H is the mean curvature, $\mathbf{I}$ is the first fundamental for, and $\mathbf{II}$ is the second fundamental form.

5) Show that a closed connected surface $X \in \mathbb{R}^3$ is a plane if and only if for each $x \in X$ there exist distinct straight lines $\ell_1, \ell_2, \ell_3 \subseteq X$ such that $\ell_1 \cap \ell_2 \cap \ell_3 = \{x\}$.

6) consider the parametrized surface $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ such that
$$\sigma(u, v) = (\cos(u) - v\sin(u), \sin(u) + v\cos(u), u + v) .$$
Calculate the Gaussian and mean curvature. Are there any asymptotic curves?

Part III: Manifolds

7) Consider the 3-dimensional *Heisenberg group* X with underlying set $\mathbb{R}^3$ and multiplication $(x, y, z)(x', y', z') = (x+x', y+y', z+z'+xy')$. Find a Riemannian metric on X that is invariant under left multiplication and calculate the corresponding Ricci tensor.

8) Let (M, g) be a Riemannian manifold. Given symmetric tensors T_1, T_2 of order 2, their *Kulkarni-Nomizu prduct* $T_1 \circ T_2$ is defined by
$$\begin{aligned}(T_1 \circ T_2)(X_1, X_2, X_3, X_4) = {} & T_1(X_1, X_3)T_2(X_2, X_4) + T_1(X_2, X_4)T_2(X_1, X_3) \\ & - T_1(X_1, X_4)T_2(X_2, X_3) - T_1(X_2, X_3)T_2(X_1, X_4) .\end{aligned}$$

Show that if (M, g) has constant curvature if and only if the Riemann curvature tensor R is a constant multiple of $g \circ g$. In the case $\dim(M) = 3$ show that R as a linear combination of the Ricci tensor and $Kg \circ g$, where K denotes the scalar curvature.

9) Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds. Consider the product space $(M_1 \times M_2, g)$, where $g((X_1, X_2), (Y_1, Y_2)) = g_1(X_1, Y_1) + g_2(X_2, Y_2)$. Calculate the Levi-Civita connection on the product in terms of the Levi-Civita connections on the two factors. Describe the geodesics of $(M_1 \times M_2, g)$ in terms of the geodesics on (M_1, γ_1) and (M_2, γ_2).