Complex Analysis Honors Exam
Spring, 2020

Note: The only integration referred to or needed is Riemann integration, either on closed intervals, $[a, b] \subset \mathbb{R}$, or its extension to improper integrals on closed half-lines, $[a, \infty)$.

1. Let $n \in \mathbb{N}$ be odd and $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, $a_n \neq 0$, an nth degree polynomial with real coefficients. Prove that $p : \mathbb{R} \to \mathbb{R}$ is surjective (onto).

2. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on an interval $[a, b] \subset \mathbb{R}$. Suppose that $\{f_n\}$ converges uniformly on $[a, b]$ as $n \to \infty$. Prove that $\lim_{n\to\infty} \int_a^b f_n(x)\, dx$ exists.

3. Let $\phi : \mathbb{Z} \to [0, \infty)$, and define a function $d : \mathbb{Z} \times \mathbb{Z} \to [0, \infty)$ by $d(n, m) = \phi(n - m)$, $\forall n, m \in \mathbb{Z}$.

   (i) Find nontrivial condition(s) on ϕ implying that d is a metric on $\mathbb{Z}$.

   (ii) For ϕ satisfying the condition(s) from (i), find additional nontrivial condition(s) implying that $(\mathbb{Z}, d)$ is a complete metric space.

4. The space of continuous functions $C[a,b]$ on a closed, bounded interval $[a,b] \subset \mathbb{R}$ is known to be a complete metric space with respect to

$$\rho(f,g) = \sup_{a \le x \le b} |f(x) - g(x)| \, .$$

Let $[a,b] \times [a,b] = \{(x,y) \mid x, y \in [a,b]\} \subset \mathbb{R}^2$, and suppose Suppose that $K \in C([a,b] \times [a,b])$ is a continuous, $\mathbb{R}$-valued function. Show that the mapping T defined by $(Tf)(x) = \int_a^b K(x,y) f(y) \, dy$ is well defined, $T : C[a,b] \to C[a,b]$, and, with respect to the metric space topology on $(C[a,b], \rho)$, is a continuous mapping, $T : C[a,b] \to C[a,b]$.

5. A function $H : \mathbb{R}^n \to \mathbb{R}$ is **Lipschitz continuous** if $\exists\, M$ such that $|H(t) - H(s)| \le M|t - s|$, for all $t, s \in \mathbb{R}^n$, $|t - s| \le 1$.

Let $f_1, f_2 \in C[a,b]$ as above, and define

$$H(t_1, t_2) = \sup_{a \le x \le b} |t_1 f_1(x) + t_2 f_2(x)|.$$

Prove that $H : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz continuous.

6. For what values of $p \in \mathbb{R}$ is the function $u(z) = |z|^p$ the real part of a holomorphic function on the punctured plane, $\mathbb{C} \setminus \{0\}$?

7. Find the number of roots (counting multiplicity) of

$$g(z) = z^6 + 2z^4 + 5z + 1$$

in the unit disc in $\mathbb{C}$.

8. Prove that the function $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} \, dt$ is holomorphic on the half-plane $\{Re(z) > 1\}$.

9. Let f be a function, holomorphic on a connected, bounded domain $U \subset \mathbb{C}$, and continuous on $\overline{U}$. Suppose $z_0 \in U$ is the only zero of f in $\overline{U}$. Find

$$\lim_{\epsilon \to 0^+} \frac{\log \left(Area \left(\{ z \in U \, : \, |f(z)| < \epsilon \} \right) \right)}{\log (\epsilon)}.$$