Swarthmore College
Department of Mathematics and Statistics
Honors Examination in Topology 2018

Instructions: Do as many of the following 12 problems as thoroughly as you can in the time you have. Include at least one problem from each of the three parts of the exam. You may use without proof the basic theorems that you have learned, but be sure to state them carefully. You may also use a result from one problem in solving another, even if you didn't solve the first problem.

Notation

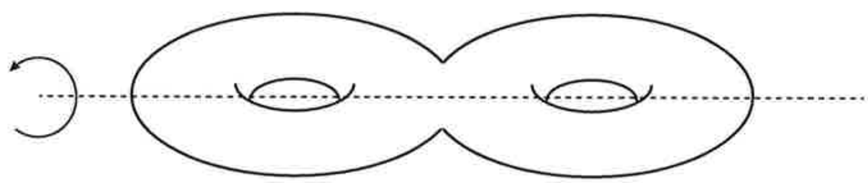
The following conventions for notation have been followed:
$\mathbb{Z}$ = the set (group, ring) of rational integers.
$\mathbb{R}$ = the set (group, field) of real numbers.
S^{n-1} = the unit sphere in $\mathbb{R}^n$.

Point Set Topology

1. Let $\mathcal{T}_M = \{U \cup F \subset \mathbb{R} \mid U$ is open in the usual topology and $F \subset \mathbb{R} \setminus \mathbb{Q}\}$. Prove that $\mathcal{T}_M$ is a topology on $\mathbb{R}$. We call $(\mathbb{R}, \mathcal{T}_M)$ the *Michael line* for E. A. Michael, its discoverer. Prove that any dense subset of the Michael line must contain all the irrational numbers. If r is an irrational real number, and $A \subset \mathbb{R}$, show that r is not a limit point of A.

2. A generalization of compactness is the Lindelöf condition: A space X is *Lindelöf* if any open cover of X has a countable subcover. Give an example of a space that is Lindelöf but not compact. Prove your assertion. Show that the continuous image of a Lindelöf space is Lindelöf. If $X = \mathbb{R}$ with the topology generated by the basis of intervals of the form $[a, b)$, the half-open topology, show that X is Lindelöf, but $X \times X$ is not Lindelöf.

3. A particularly useful result concerns a continuous bijection $f\colon X \to Y$ where X is compact and Y is Hausdorff. The conclusion is that f is a homeomorphism. Prove this result. Now let's show that all the assumptions are needed. Give an example of a continuous bijection $f\colon X \to Y$ which is not a homeomorphism for X compact and Y non-Hausdorff, and for X noncompact with Y Hausdorff.



4. Consider a nice symmetric version of the double torus T, as pictured. If we rotate the space through π radians, we map T to itself. If we identify points that are rotated to one another, what quotient space do we obtain? Indicate in your description which points in the quotient have different cardinalities of their preimage in the quotient map.

5. The fundamental group can restrict the retracts that a space can have. Recall a subspace $A \subset X$ is a *retract* if there is a continuous mapping $r\colon X \to A$ for which the composite $A \hookrightarrow X \xrightarrow{r} A$ is the identity map on A. Prove that if A is a retract of X and $a_0 \in A$, then the induced mapping $i_\#\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ is injective and $r_\#\colon \pi_1(X, a_0) \to \pi_1(A, a_0)$ is surjective. Suppose you know that $\pi_1(X, a_0) \cong \mathbb{Z}/p\mathbb{Z}$ where p is a prime number. Then prove that the only retracts of X are subspaces that are either simply connected or with fundamental group isomorphic to $\pi_1(X, a_0)$. What would be a generalization of this result?

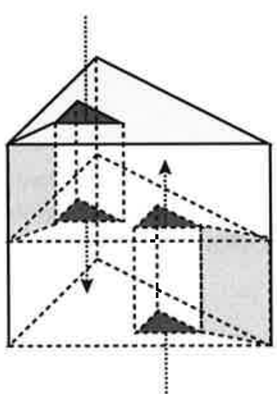
6. A topological group is a space G that is also a group for which the group multiplication $\mu\colon G \times G \to G$ and inverse map $\nu\colon G \to G$, $\nu(g) = g^{-1}$ are continuous mappings. For example, the space of invertible 2×2 real matrices, as a subspace of $\mathbb{R}^4$, with matrix multiplication is a topological group. A *path component* of G is an equivalence class $[g]$ under the relation $g \sim h$ if there is a path joining g to h. Prove that the set of path components, $\pi_0(G)$, is also a group with operation $[g] \cdot [h] = [gh]$. Show that the set of group elements in the path component of the identity element $[e] = \{h \in G \mid e \sim h\}$ forms a normal subgroup of G.

7. Prove that the connected sum of three projective planes is homeomorphic to the connected sum of a torus and a projective plane. Describe the fundamental groups of each representation in terms of generators and relations. Of course, these determine the same group. Is there an algebraic way to see that these are isomorphic groups? Discuss.

8. A mapping of pointed spaces $p\colon (E, e_0) \to (B, b_0)$ has the *homotopy lifting property* (HLP) with respect to a pointed space (X, x_0) if for any continuous mapping $g\colon (X, x_0) \to (E, e_0)$ and a based homotopy $H\colon (X, x_0) \times I \to (B, b_0)$ for which $H(x, 0) = p \circ g(x)$ and $H(x_0, t) = b_0$ for all $t \in I$, there is a based homotopy $\hat{H}\colon (X, x_0) \times I \to (E, e_0)$ with $\hat{H}(x, 0) = g(x)$ and $\hat{H}(x_0, t) = e_0$ for all t. Let $F = p^{-1}(b_0)$, a subspace of E with $e_0 \in F$. Prove that, if p has the HLP with respect to $(S^1, 1)$, then the sequence of fundamental groups

$$\pi_1(F, e_0) \xrightarrow{i_\#} \pi_1(E, e_0) \xrightarrow{p_\#} \pi_1(B, b_0),$$

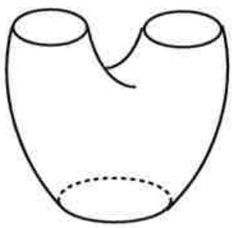
is exact at $\pi_1(E, e_0)$, that is, $\ker p_\# = \operatorname{im} i_\#$.



9. The object pictured is a simplicial 2-complex with two 'rooms', the entries to which are from above and below as pictured by the arrowed lines. Essentially the two rooms are stacked over one another, joined to the outside by triangular tubes that are joined by a panel to a wall. If you are standing inside the upper room, there is a triangular shaped column inside the room attached to the corner of the room by a panel, and there is a triangular shaped hole in the floor that passes to the outside. The darkest shaded triangles are not part of the complex. What is the Euler characteristic of this complex? Explain your computation. (You can assume the relation $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$ that follows from the Mayer-Vietoris sequence, if it helps.)

10. Suppose Π is a finite polyhedron that is homeomorphic to the sphere, S^2. Suppose V denotes the number of vertices, E, the number of edges, and F, the number of faces making up the polyhedron. Each face must have at least 3 edges, and each vertex belongs to at least 3 edges. Let V_i denote the number of vertices belonging to exactly i edges, and F_j, the number of faces possessing exactly j edges. Hence $V = V_3 + V_4 + V_5 + \cdots$ and $F = F_3 + F_4 + F_5 + \cdots$. Prove the following relations:
(a) $E \neq 7$;
(b) There are at least 4 faces with fewer than 6 edges, that is, $F_3 + F_4 + F_5 \geq 4$;
(c) $V_3 + V_5 + V_7 + \cdots$ is even.

11. Suppose $h\colon \mathcal{T}op \to \mathcal{A}b$ is a functor on the category of topological spaces to the category of abelian groups. Suppose that X is a space for which $h(X) \cong \mathbb{Z}$ and A is a subspace of X with $h(A) \cong \mathbb{Z}/2\mathbb{Z}$. Prove that A is *not* a retract of X.



12. A useful space in studying surfaces is P, a pair of pants. Compute the homology of P. Explain in detail how you did your computation.