SWARTHMORE COLLEGE HONORS EXAM 2018
REAL ANALYSIS

Instructions: Do as many problems, or parts of problems, or special cases of problems, as you can. Justify all answers. You may quote any standard result as long as that result is not essentially what you are being asked to prove.

Real analysis I

1. Prove or disprove the following statements. If a statement is false, what additional hypotheses would make it true? (Given a function $f : X \to X$, the function $f^{\circ 2} = f \circ f$ is the composition of f with itself, that is, $f^{\circ 2}(x) = f(f(x))$.)

(a) Let $f : [0,1] \to [0,1]$ be a continuous function.
 (i) If f is differentiable, then so is $f^{\circ 2}$.
 (ii) If $f^{\circ 2}$ is differentiable, then so is f.

(b) Let $f : [0,1] \to [0,1]$ be a function (not necessarily continuous).
 (i) If f is Riemann integrable, then so is $f^{\circ 2}$.
 (ii) If $f^{\circ 2}$ is Riemann integrable, then so is f.

2. (a) State and prove the Mean Value Theorem.
(b) Use the Mean Value Theorem to prove the following weak form of the Fundamental Theorem of Algebra: *A real polynomial of degree n has at most n (real) roots.*

3. Define a sequence of functions $f_1, f_2, \ldots : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} n^4x^2 - n^3x, & 0 \le x \le \frac{1}{n} \\ 0, & \frac{1}{n} < x \le 1. \end{cases}$$

Discuss the convergence of $\{f_n\}$, $\{f_n'\}$, and $\{\int_0^1 f_n(x)\,dx\}$ as $n \to \infty$. If the convergence isn't nice, explain what additional hypotheses would make it nice.

4. Let $C^\infty[0,1]$ be the set of infinitely differentiable functions from the interval $[0,1]$ to $\mathbb{R}$. We define two metrics on $C^\infty[0,1]$ as follows:

- $d_0(f,g) = \max\limits_{0 \le x \le 1} |f(x) - g(x)|$
- $d_1(f,g) = \max(d_0(f,g), d_0(f',g'))$

(a) Consider the map $D : C^\infty[0,1] \to C^\infty[0,1]$ which sends a function to its derivative, that is, $D(f) = f'$. Is D continuous considered as a map from $(C^\infty[0,1], d_0)$ to $(C^\infty[0,1], d_0)$? What about considered as a map from $(C^\infty[0,1], d_1)$ to $(C^\infty[0,1], d_0)$?

(b) Is the set of polynomials open in $(C^\infty[0,1], d_0)$? Is it closed?

(c) Is the space $(C^\infty[0,1], d_0)$ complete?

5. The Euler dilogarithm function $L(x)$ is given by $L(x) = \displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k^2}$. Where is $L(x)$ defined? Where is it continuous? (Can you compute the values of $L(x)$ for any particular x's?)

6. The Hénon map $H_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ is an important example in dynamical systems. It is given by $H_{a,b}(x,y) = (y, 1 - ay^2 + bx)$, where a and b are parameters. A point (x_0, y_0) is a *fixed point* for $H_{a,b}$ if $H_{a,b}(x_0, y_0) = (x_0, y_0)$.

(a) What conditions on a, b, x_0, and y_0 will ensure that the coordinates of the fixed points are differentiable functions of the parameters a and b?

(b) If $a = 6$ and $b = 0.9$, then the point $(0.4, 0.4)$ is fixed.

(i) What will $H_{6,0.9}$ do to points (x,y) that are near $(0.4, 0.4)$?

(ii) What will happen to the coordinates of that fixed point if a is increased slightly, holding b fixed?

7. Define M_n to be the set of all finite line segments in $\mathbb{R}^n$, that is, $M_n = \left\{\overline{\mathbf{AB}} : \mathbf{A}, \mathbf{B} \in \mathbb{R}^n, \mathbf{A} \neq \mathbf{B}\right\}$ (we consider the line segments $\overline{\mathbf{AB}}$ and $\overline{\mathbf{BA}}$ to be identical).

(a) Put a manifold structure on M_1. What is the dimension of M_1? Is it orientable?

(b) Same question for M_3.

8. Define an $(n-1)$-form ω by $\omega = (x_1 - x_2^2 + x_3^3 - \cdots \pm x_n^n)\left(\sum_{1=1}^{n} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\right)$. Let C be the n-dimensional cube given by $0 \le x_i \le 2$, $1 \le i \le n$. For what values of n will the integral of w over the boundary of C be between 100 and 1000?

9. Let $\int dt$ represent the Riemann integral and $\int d\lambda$ the Lebesgue integral (both on $\mathbb{R}$). Let f be a bounded function on the compact interval $[a,b]$.

(a) Show that if $\int_a^b f(t)\,dt$ exists, then so does $\int_{[a,b]} f\,d\lambda$. What additional conditions are necessary for the two integrals to be equal?

(b) The Fundamental Theorem of Calculus tells us that if f is continuous, then $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$. Show that this is not true for measurable functions and the Lebesgue integral. More precisely: give an example of a measurable f such that $\frac{d}{dx}\int_{[a,x]} f\,d\lambda \neq f(x)$, even though $\frac{d}{dx}\int_{[a,x]} f\,d\lambda$ exists and is continuous.

10. (a) Use the generalized Stokes theorem to prove the divergence theorem:
Let M be a 3-manifold in $\mathbb{R}^3$ and F a smooth vector field defined in a neighborhood of V. Then $\iiint_M (\nabla \cdot F)\,dV = \iint_{\partial M} (F \cdot \vec{n})\,dA$, where $\vec{n}$ is an outward-pointing normal vector.

(b) Put coordinates ρ, θ, ϕ on $\mathbb{R}^3$, with $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho\cos\phi$. Find dx, dy, dz, and $dx \wedge dy \wedge dz$ in terms of $d\rho$, $d\theta$, and $d\phi$.