Swarthmore College
Department of Mathematics and Statistics
Honors Examination in Geometry 2018

Instructions: Do as many of the following 12 problems as thoroughly as you can in the time you have. Try to include at least one problem from each of the four parts of the exam. You may use without proof the basic theorems that you have learned, but be sure to state them carefully. You may also use a result from one problem in solving another, even if you didn't solve the first problem. If you have lots of time left over, feel free to solve other problems.

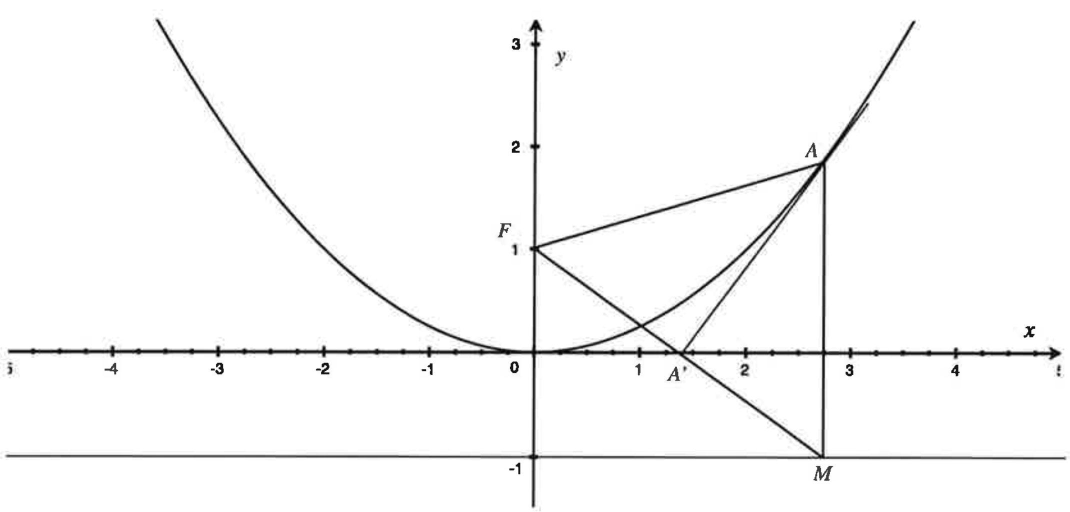
NOTATION

The following conventions for notation have been followed:

$\mathbb{R}$ = the set (group, field) of real numbers.
$\mathbb{C}$ = the set (group, field) of complex numbers.
S^{n-1} = the unit sphere in $\mathbb{R}^n$.

CURVES

1. A parabola is defined as the curve generated by points (x, y) that are equidistant from a point F (the focus) and a line (the directrix). If the focus is the point (0,1) and the directrix is the line $y = -1$, give a parametrization of the parabola. What is arc length function along the curve? If AA' is a tangent to the parabola, with A' on the line between the focus F and the point M with AM perpendicular to the directrix, prove that A' lies on the x-axis and $\angle AA'F$ is a right angle.



2. Suppose a piece of plywood is carved with the outline of the parabola in problem 1. If you roll the parabola without slippage along the x-axis, give a parametrization of the curve made by the focus as you roll the shape. (Hint: Problem 1 can help.)

3. The Frenet frame $\{T(s), N(s), B(s)\}$ along a unit speed curve in $\mathbb{R}^3$ is an orthonormal frame and so any vector $\mathbf{v}$ may be written $\mathbf{v} = aT(s) + bN(s) + cB(s)$ at $\alpha(s)$. Show that there is a vector D at $\alpha(s)$ with the property that $D \times T = T'$, $D \times N = N'$ and $D \times B = B'$ where we are taking the derivative with respect to s of the vectors in the Frenet frame. Show that the matrix that expresses the Frenet-Serret theorem has rank 2, and that D is in the null space of the matrix.

4. The wheels of a bicycle of unit length leaves two tracks in the dirt as the front wheel follows a unit speed curve $\alpha(s)$ in the plane. If we denote the position of the front wheel by the curve $\alpha(s)$ and the position of the back wheel by $\beta(s)$, then the frame of the bike is always tangent to the curve $\beta(s)$. Let $\theta(s)$ denote the angle between $\alpha'(s)$ and $\alpha(s)-\beta(s)$ (the frame). Then $\cos\theta(s) = (\alpha(s)-\beta(s))\cdot\alpha'(s)$. Use the planar frame $\{T, N\}$ along $\alpha(s)$ to deduce the following differential equation for $\theta(s)$:

$$\theta'(s) - \kappa(s) = -\sin\theta(s),$$

where $\kappa(s)$ is the planar curvature of $\alpha(s)$.

Surfaces, curvature

The *helicoid* is a surface H that can be parametrized

$$X\colon (-\pi,\pi)\times(0,2\pi)\to\mathbb{R}^3,\quad X(u,v) = (v\cos u, v\sin u, u).$$

Each fixed value of v determines a helix in $\mathbb{R}^3$.

5. Compute the differential geometric apparatus associated to the helicoid, that is, the coefficients of the first fundamental form, E, F, and G, the associated normal to H, and the coefficients of the second fundamental form e, f, and g (L, M, and N, if you prefer).

6. With your apparatus, compute the Gaussian curvature K on H. For a triangle made up of geodesics on H, the sum of the interior angles of the triangle satisfies what property? Give a proof of your answer.

7. The helices for a choice of v are coordinate curves. Holding one of u or v constant and varying the other determines the coordinate curves. Are these curves orthogonal on H? Are either of the coordinate curves geodesics on H? Explain your answer.

Geometry on surfaces

8. A great circle is a geodesic on the round sphere $S^2 \subset \mathbb{R}^3$. Give a convincing proof of this fact, which is often assumed in discussions of spherical geometry.

9. If you have a surface in $\mathbb{R}^3$ and you want to wrap a cloth on it, you want the cloth to lie nicely, that is, the threads aren't stretched, though threads may meet nonorthogonally. If this cloth were a coordinate chart, then it would be called a *Tchebyshev net*, and it would determine a metric of the form

$$ds^2 = du^2 + 2F\,du\,dv + dv^2.$$

Thus the coordinate curves are unit speed along the surface. At a point p, let $\omega(p)$ denote the angle between the coordinate curves through p in this patch. Show that $\cos\omega(p) = F(p)$, and that the Gaussian curvature of the surface is $K = \dfrac{-\omega_{uv}}{\sin\omega}$.

10. Suppose S is a surface in $\mathbb{R}^3$ that is homeomorphic to the sphere S^2. Suppose further that γ is a closed and simple (no double points) geodesic on S. On the round sphere S^2, this would be a great circle which divides S^2 into two parts of equal area. Prove that γ divides S into two parts of equal total Gaussian curvature, where the total Gaussian curvature over a region R is given by $\iint_R K\,dA$.

Geometry more generally

11. The Poincaré half-plane is the abstract surface with $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ as chart and Riemannian metric given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

View $\mathbb{H}$ as a subset of the complex plane and denote (x, y) in $\mathbb{H}$ as $z = x + iy$. Show that the set of real linear fractional transformations

$$z \mapsto \frac{az + b}{cz + d}, \text{ where } a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1,$$

determine isometries of the Poincaré half-plane.

12. Locally, a coordinate chart $X\colon (a, b) \times (c, d) \to \mathbb{R}^3$ can be replaced with an abstract surface $U = (a, b) \times (c, d)$ with a metric $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$. What geometric features of the embedded surface can be determined only from (U, ds^2)? That is, what is enough structure to do geometry? Discuss these questions in a short essay.

Some formulas

Brioschi's formula for curvature:

$$K = \frac{1}{(EG - F^2)^2}\left(\det\begin{bmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{bmatrix} - \det\begin{bmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{bmatrix}\right).$$

$$\Gamma_{ij}^m = \frac{1}{2}\sum_k \left\{\frac{\partial}{\partial u_i} g_{jk} + \frac{\partial}{\partial u_j} g_{ki} - \frac{\partial}{\partial u_k} g_{ij}\right\} g^{km}.$$

$$\Gamma_{11}^1 = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, \qquad \Gamma_{11}^2 = \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)},$$

$$\Gamma_{12}^1 = \frac{GE_v - FG_u}{2(EG - F^2)}, \qquad \Gamma_{11}^2 = \frac{EG_u - FE_v}{2(EG - F^2)},$$

$$\Gamma_{22}^1 = \frac{2GF_v - GG_u + FG_v}{2(EG - F^2)}, \qquad \Gamma_{22}^2 = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}.$$