Swarthmore College
Department of Mathematics & Statistics
Algebra Honors Examination
Spring 2018

This exam contains nine problems. Try to solve *six* problems as completely as possible. Once you are satisfied with your responses to six problems, make a second pass through the exam and complete as many parts of the remaining problems as possible. I am interested in your thoughts on a problem and attempts at special cases, even if you do not completely solve the problem. When there are multiple parts, you are permitted to address a later part without solving all the earlier ones. Please submit your solution even if you cannot do all the parts of a problem, or even if you cannot solve the problem in the full level of generality requested.

1. Let p be a prime, and define the *Heisenberg group* to be the set

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{F}_p \right\}$$

under the operation of matrix multiplication.

(a) Find the center $Z(H)$ of H.

(b) Prove that $H/Z(H)$ is abelian.

(c) All finite abelian groups are isomorphic a group of a certain familiar form. Which one is $H/Z(H)$?

2. Choose <u>one</u> of the following two problems about the symmetric group S_n.

(a) State and prove a necessary and sufficient condition so that the n-cycle $(1\ 2\ \cdots\ n)$ and the transposition $(a\ b)$ generate S_n. (Your condition will depend on a, b, and n.)

(b) Prove that every subgroup H of S_n such that $[S_n : H] = n$ is isomorphic to S_{n-1}.

3. Let H and K be subgroups of a group G. Given $x \in G$, define the HK-*double coset* of x as

$$HxK = \{hxk \mid h \in H, k \in K\}.$$

Note that HxK is the orbit of xK under the action of H on G/K by left multiplication.

(a) Prove that the set of HK-double cosets partitions G.

(b) Prove that $|HxK| = |K| \cdot |H : H \cap xKx^{-1}|$.

(c) Do all HK-double cosets have the same size? If yes, prove it; if no, provide an example.

4. Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$, where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$ be the group of *quaternions*.

(a) Find a presentation for Q using only two generators.

(b) Find the character table for all irreducible complex representations of Q.

(c) Which (if any) among these representations are faithful?

5. Define the complex vector space $\mathfrak{sl}_2(\mathbb{C})$ to be the set of 2×2 trace zero matrices. Define a linear operator $T : \mathfrak{sl}_2(\mathbb{C}) \longrightarrow \mathfrak{sl}_2(\mathbb{C})$ by $T(A) = A + A^t$, where A^t denotes the transpose.

(a) Find a basis for $\mathrm{Im}(T)$ and $\mathrm{Ker}(T)$.

(b) If you haven't already in (a), now find a basis for $\mathfrak{sl}_2(\mathbb{C})$ consisting of eigenvectors.

(c) Write down the matrix for T corresponding to both your bases from (a) and (b), and explain the relationship between these two matrices.

(d) Is $\mathfrak{sl}_2(\mathbb{C}) \cong \mathrm{Im}(T) \oplus \mathrm{Ker}(T)$? What does this mean geometrically?

6. An element $a \in \mathbb{Z}/n\mathbb{Z}$ is *nilpotent* if there exists a nonnegative integer k such that $a^k = 0$.

(a) Prove that the nilpotent elements of $\mathbb{Z}/n\mathbb{Z}$ form a subring.

(b) Which elements of $\mathbb{Z}/n\mathbb{Z}$ are nilpotent? Under what hypotheses does there exist a *nonzero* nilpotent element in $\mathbb{Z}/n\mathbb{Z}$?

(c) Which elements of $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ are nilpotent?

7. Let R be the ring of rational numbers of the form $\dfrac{a}{b}$ where b is odd.

(a) Prove that R is an integral domain.

(b) Denote by $U(R)$ the group of units. Prove that

$$M = R \backslash U(R) = \left\{ \frac{a}{b} \in R \;\middle|\; \frac{a}{b} \notin U(R) \right\}$$

is a maximal ideal in R.

(c) Find all primes in R.

(d) Is R a PID and/or UFD?

8. Denote by $\mathbb{F}_3$ the finite field containing 3 elements. (Note that below we suppress the coset notation and simply write x for $\overline{x} = x + \langle f(x) \rangle$; feel free to do the same as long as it's clear from context what you mean.)

(a) Is $x^2 + x + 1$ a unit in $\mathbb{F}_3[x]/\langle x^2 + 1 \rangle$?

(b) For which integers a is $\mathbb{F}_3[x]/\langle x^3 + a \rangle$ a field?

(c) Construct a finite field which contains *all* of the roots of the polynomial $x^3 + x + 1$.

9. Consider the polynomial $f(x) = x^4 - x^2 - 6$.

(a) Find the splitting field K for $f(x)$ over $\mathbb{Q}$.

(b) Find the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$.

(c) Draw the subgroup lattice for G and the corresponding subfield lattice for K.

(d) Choose one intermediate subfield L of K and denote its corresponding subgroup by H. According to the Fundamental Theorem of Galois Theory, H is the Galois group of which field extension? Verify this fact in your chosen example.

(e) Choose one normal subgroup N of G and denote its corresponding subfield by F. According to the Fundamental Theorem of Galois Theory, $\mathrm{Gal}(F/\mathbb{Q})$ is isomorphic to which group? Verify this fact in your chosen example.