Swarthmore College
Department of Mathematics and Statistics
Honors Examination in Topology 2017

Instructions: Do as many of the following 12 problems as thoroughly as you can in the time you have. Include at least one problem from each of the three parts of the exam. You may use without proof the basic theorems that you have learned, but be sure to state them carefully. You may also use a result from one problem in solving another, even if you didn't solve the first problem. If you have lots of time left over, feel free to solve other problems.

Notation

The following conventions for notation have been followed:
$\mathbb{Z}$ = the set (group, ring) of rational integers.
$\mathbb{Q}$ = the set (group, field) of rational numbers.
$\mathbb{R}$ = the set (group, field) of real numbers.
$\mathbb{C}$ = the set (group, field) of complex numbers.
S^{n-1} = the unit sphere in $\mathbb{R}^n$.

Point Set Topology

1. Often among the minimal requirements of a topological space we find the Hausdorff condition. Write an essay (around a page long) on what the Hausdorff condition provides. What would it be like if absent?

2. Given two spaces X and Y and a mapping $f\colon X \to Y$ we can form the *mapping cone* of f which is the quotient space of $(X \times [0,1]) \amalg Y$ (disjoint union) identifying $(x,1) \in X \times [0,1]$ with $f(x)$ in Y and $(x,0)$ with $(x',0)$ in $X \times [0,1]$ for any x, x' in X. We denote this quotient by Cone(f). Suppose $f\colon S^1 \to S^1$ is the squaring map, $f(e^{i\theta}) = e^{2i\theta}$. Show that the space Cone(f) is homeomorphic to $\mathbb{R}P^2$, the real projective plane.

3. Suppose X is a compact Hausdorff space. Let $f\colon X \to X$ be a continuous self-map. Prove that there is a closed and nonempty subset $B \subset X$ for which $f(B) = B$. Why do we make these assumptions? If you lift any of the assumptions, show how this property can fail. Does the conclusion imply there is a fixed point of the mapping f?

4. Let Y denote a subset of $\mathbb{R}^3$ that is a Möbius band without its boundary. You can think of this space as an open interval centered at each point on a circle, twisting around to give the Möbius band shape. Or you can make a quotient of $[0,1] \times (0,1)$ identifying the subspaces $\{0\} \times (0,1)$ and $\{1\} \times (0,1)$ by $0 \times t \sim 1 \times 1 - t$. Show that Y is locally compact but not compact and construct the one-point compactification of Y, $\hat{Y} = Y \cup \{\infty\}$. Is $\hat{Y}$ a familiar space? Prove your answer.

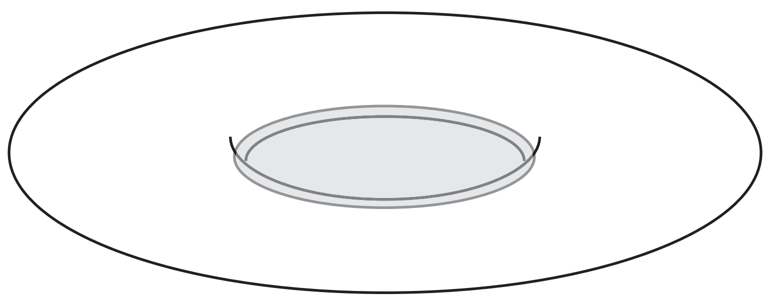
5. For $p < n$, let $\overline{\mathbb{R}}^p$ denote the homeomorphic copy of Euclidean p-space $\mathbb{R}^p$ lying in $\mathbb{R}^n$ given by

$$\overline{\mathbb{R}}^p = \{(a_1, a_2, \ldots, a_p, 0, 0, \ldots, 0) \in \mathbb{R}^n \mid a_i \in \mathbb{R}\}.$$

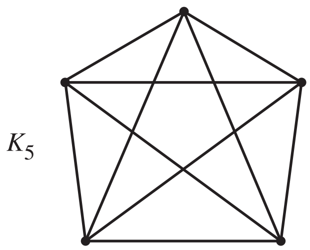
Show that $\mathbb{R}^n - \overline{\mathbb{R}}^p$ is homotopy equivalent to S^{n-p-1}.

6. The inclusion of $i\colon \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ by $i(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0)$ induces an inclusion of spheres $S^{n-1} \subset S^n$. We can think of extending these inclusions infinitely often as subspaces of $\mathbb{R}^\infty$, the countable product of copies of the real numbers. Let S^∞ denote the union of all of the nested copies of S^n for $n \geq 0$. Show that S^∞ is not compact. (Show that the sequence $\mathbf{v}_n = (0, 0, \ldots, 1) \in S^n$ does not have a convergent subsequence.) Show that any mapping of a finite complex K into S^∞ is homotopic to a constant mapping, that is, the set of all homotopy classes of mappings $[K, S^\infty] = \{*\}$. (Hint: the compactness of K is important. Also, the image of $i(S^n) \subset S^{n+1}$ is contractible. Why?)

7. Let X be a standard torus in $\mathbb{R}^3$ together with a membrane that covers the outer hole as pictured below (if the torus is an inner tube, with the membrane you have an ideal transport for tubing). Choose a point x_0 on both the membrane and the torus. Compute $\pi_1(X, x_0)$. (I suggest a nice Seifert-van Kampen argurment.)

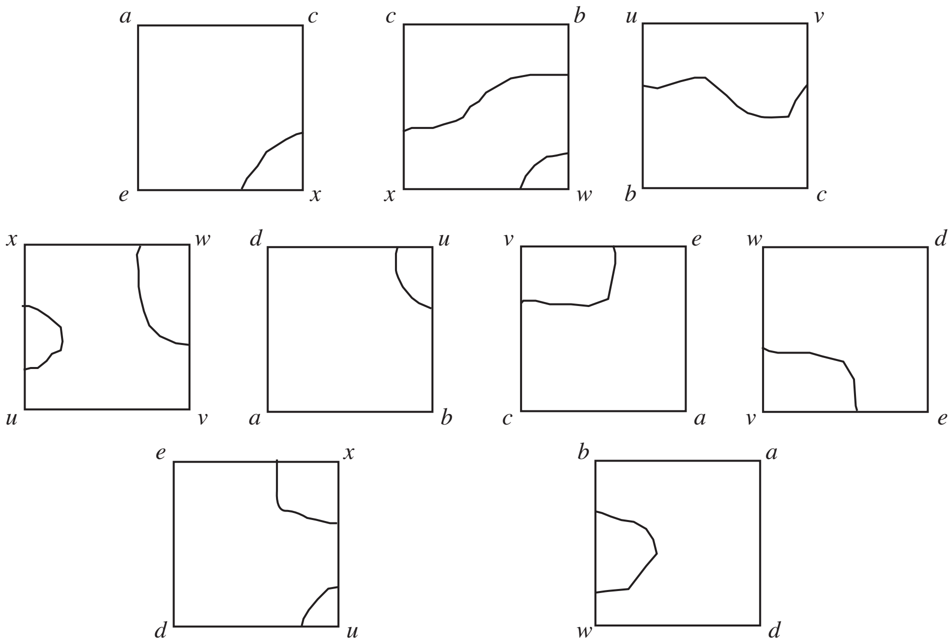


8. Suppose p is a prime number and B is a path-connected space. Suppose $b_0 \in B$ and $\pi_1(B, b_0)$ is a cyclic group of order p. If $f\colon E \to B$ is a covering space, then show that either E is simply-connected, or f is a homeomorphism.

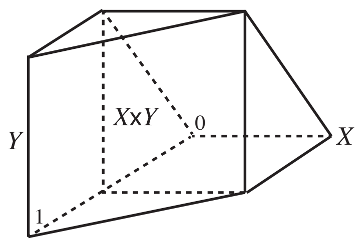


9. One of the nifty applications of the Euler characteristic is to prove that the complete graph on five vertices, K_5, that is, each vertex is connected to all of the other vertices, does not embed in the plane $\mathbb{R}^2$. Prove this. Show, however, that K_5 embeds in the Möbius band. Deduce that there is no continuous mapping of the Möbius band to the plane that is one-one.

10. When the spaceship left Earth, a stash of papers were found that was interpreted as a map of the world of the aliens. The nine constituent maps are pictured here:



Kathryn, a topologist, examined the maps and she announced, "They were from a different dimension." Since you are a topologist, explain what Kathryn meant by her outburst, and prove why you are correct.



11. The *join* operation described for simplicial complexes can be extended to all spaces as a quotient. Let $I = [0,1]$, X and Y spaces. The space $X * Y$ is the quotient of $X \times I \times Y$ by the relation $(x,0,y) \sim (x,0,y')$ and $(x,1,y) \sim (x',1,y)$ for all x, $x' \in X$ and y, $y' \in Y$. It can be pictured as shown. Alternatively, if $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$, then $X * Y \subset \mathbb{R}^{n+m}$ is the collection

$$\{((1-t)x_1, (1-t)x_2, \ldots, (1-t)x_m, ty_1, ty_2, \ldots, ty_n) \mid 0 \leq t \leq 1\}.$$

Use this to prove that $S^m * S^n \cong S^{m+n+1}$. We can use this fact and the Mayer-Vietoris sequence to prove that $H_{m+n}(S^m \times S^n) \cong \mathbb{Z}$. Recall that the Mayer-Vietoris sequence associated to a pair of open sets U and V with $U \cup V = Z$, is the long exact sequence given by

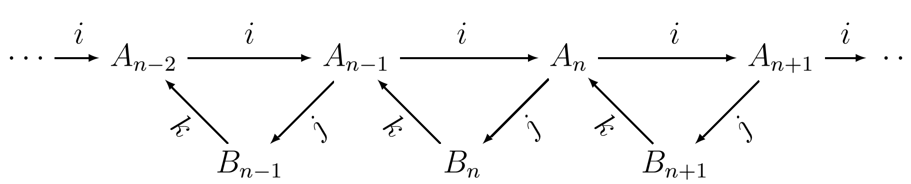
$$\cdots \to H_k(U \cap V) \xrightarrow{i_U \oplus i_V} H_k(U) \oplus H_k(V) \xrightarrow{i_U - i_V} H_k(Z) \xrightarrow{\delta} H_{k-1}(U \cap V) \to \cdots.$$

For the join, consider the subset U to be the quotient of $X \times [0, 3/4) \times Y$ and V the quotient of $X \times (1/4, 1] \times Y$. Show that U is homotopy equivalent to X, V is homotopy equivalent to Y and $U \cap V$ is homotopy equivalent to $X \times Y$. Apply this with the Mayer-Vietoris sequence to show $H_{m+n}(S^m \times S^n) \cong \mathbb{Z}$.

12. Some homological algebra: Suppose $\{A_n \mid n = 0, 1, 2, \ldots\}$ and $\{B_n \mid n = 0, 1, 2, \ldots\}$ are families of abelian groups. Suppose for each n, there is a long exact sequence

$$\cdots \xrightarrow{k} A_{n-1} \xrightarrow{i} A_n \xrightarrow{j} B_n \xrightarrow{k} A_{n-1} \xrightarrow{i} A_n \xrightarrow{j} \cdots.$$

Such a collection of exact sequences can be braided together to make a diagram of groups:



Show that the composite $d = j \circ k \colon B_n \to B_{n-1}$ is a differential, in other words, $d \circ d = 0$. It follows you can form a chain complex $B_* = (B_n, d)$, and compute the homology $H_n(B_*, d)$. Prove that $H_n(B_*, d) \cong k^{-1}(i(A_{n-2}))/j(\ker i \colon A_n \to A_{n+1})$ for each index n.