SWARTHMORE COLLEGE HONORS EXAM 2017
REAL ANALYSIS

Instructions: Do as many problems, or parts of problems, or special cases of problems, as you can. Justify all answers. You may quote any standard result as long as that result is not essentially what you are being asked to prove.

Real analysis I

1. Prove or disprove the following statements. If a statement is false, what additional hypotheses would make it true? (Given a function f, the function f^2 is defined by $f^2(x) = (f(x))^2$.)

(a) Let $f : [0,1] \to \mathbb{R}$ be a continuous function.

(i) If f is differentiable, then so is f^2.

(ii) If f^2 is differentiable, then so is f.

(b) Let $f : [0,1] \to \mathbb{R}$ be a function (not necessarily continuous).

(i) If f is Riemann integrable, then so is f^2.

(ii) If f^2 is Riemann integrable, then so is f.

2. The arithmetic mean of two non-negative real numbers x and y is $A(x,y) = \dfrac{x+y}{2}$, and their geometric mean is $G(x,y) = \sqrt{xy}$.

(a) Show that $A(x,y) \geq G(x,y)$. Under what conditions do we have equality?

(b) Given x_0, y_0, define $x_1 = A(x_0, y_0)$ and $y_1 = G(x_0, y_0)$, and, inductively, $x_n = A(x_{n-1}, y_{n-1})$ and $y_n = G(x_{n-1}, y_{n-1})$. Show that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

3. (a) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that the derivative f' exists everywhere but f' is not continuous.

(b) Show that if a function $g : \mathbb{R} \to \mathbb{R}$ is differentiable, then g' satisfies the conclusion of the Intermediate Value Theorem, even though g' may not be continuous.

4. Let S be the space of all infinite sequences of 0's and 1's, that is, $S = \{(s_1, s_2, \ldots) : s_i \in \{0,1\}\}$. Define a metric d on S by setting

$$d((s_1, s_2, \ldots), (t_1, t_2, \ldots)) = \begin{cases} \frac{1}{n} & \text{if } s_n \neq t_n \text{ and } s_i = t_i \text{ for } 1 \leq i < n, \\ 0 & \text{if } s_i = t_i \text{ for all } i. \end{cases}$$

(That is, if two sequences first differ at the tenth spot, then the distance between them is $1/10$.)

(a) Show that d really is a metric.

(b) Is the map $\sigma : S \to S$ given by $\sigma(s_1, s_2, \ldots) = (s_2, s_3, \ldots)$ continuous?

(c) Are the following subsets of S open? Closed?

(i) The set A of all sequences that contain exactly two 1's.

(ii) The set B of all sequences that begin with a 0.

5. Define a sequence of functions $f_1,\ f_2, \ldots : [0,1] \to \mathbb{R}$ by $f_n(x) = \dfrac{nx^n}{1+nx^n}$. Discuss the convergence of $\{f_n\}$, $\{f_n'\}$, and $\{\int_0^1 f_n(x)\,dx\}$ as $n \to \infty$. If the convergence isn't nice, explain what additional hypotheses would make it nice.

6. Let M be a compact, connected, orientable n-manifold without boundary, and let ω be an $(n-1)$-form. Show that $d\omega$ must vanish at some point of M.

7. Economists often assume that a system of equations such as

$$x^2 - y^2 - u^3 + v^2 + 4 = 0$$
$$2xy + y^2 - 2u^2 + 3v^4 + 8 = 0$$

defines u and v as functions of x and y. They further assume that a small change in x and y leads to a small change in u and v.

(a) Under what conditions, if any, are these assumptions justified for this particular system?

(b) One solution is $x = 2$, $y = -1$, $u = 2$, $v = 1$. What will happen to u if y is increased slightly, holding x fixed?

8. A smooth function f is an *integrating factor* for a 1-form ω if f is never 0 and $f\omega$ is exact.

(a) Show that if ω has an integrating factor, then $\omega \wedge d\omega = 0$ and there is a 1-form α such that $d\omega = \alpha \wedge \omega$.

(b) Let $F(\mathbf{x}) = (p(\mathbf{x}), q(\mathbf{x}), r(\mathbf{x}))$ be a smooth vector field on $\mathbb{R}^3$. Show that if the 1-form $p\,dx + q\,dy + r\,dz$ has an integrating factor, then the curl of F is orthogonal to F everywhere.

9. Let $f : [0,1] \to \mathbb{R}$ be a function, and define the function f^2 by $f^2(x) = (f(x))^2$. Prove or disprove the following statements. If a statement is false, what additional hypotheses would make it true?

(a) If f is Lebesgue integrable, then so is f^2.

(b) If f^2 is Lebesgue integrable, then so is f.

(c) If f is measurable, then so is f^2.

(d) If f^2 is measurable, then so is f.

10. Identify the set of 2×2 matrices with $\mathbb{R}^4$. Show that the subset of matrices of rank 1 is a manifold. (Recall that a 2×2 matrix has rank 1 if and only if at least one of its rows is nonzero and the other row is a constant multiple of the nonzero row.)