SWARTHMORE COLLEGE HONORS EXAM 2017
COMPLEX ANALYSIS

Instructions: Do as many problems, or parts of problems, or special cases of problems, as you can. Justify all answers. You may quote any standard result as long as that result is not essentially what you are being asked to prove.

Real analysis I

1. Prove or disprove the following statements. If a statement is false, what additional hypotheses would make it true? (Given a function f, the function f^2 is defined by $f^2(x) = (f(x))^2$.)

(a) Let $f : [0,1] \to \mathbb{R}$ be a continuous function.
  (i) If f is differentiable, then so is f^2.
  (ii) If f^2 is differentiable, then so is f.

(b) Let $f : [0,1] \to \mathbb{R}$ be a function (not necessarily continuous).
  (i) If f is Riemann integrable, then so is f^2.
  (ii) If f^2 is Riemann integrable, then so is f.

2. The arithmetic mean of two non-negative real numbers x and y is $A(x,y) = \dfrac{x+y}{2}$, and their geometric mean is $G(x,y) = \sqrt{xy}$.

(a) Show that $A(x,y) \geq G(x,y)$. Under what conditions do we have equality?

(b) Given x_0, y_0, define $x_1 = A(x_0, y_0)$ and $y_1 = G(x_0, y_0)$, and, inductively, $x_n = A(x_{n-1}, y_{n-1})$ and $y_n = G(x_{n-1}, y_{n-1})$. Show that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

3. (a) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that the derivative f' exists everywhere but f' is not continuous.

(b) Show that if a function $g : \mathbb{R} \to \mathbb{R}$ is differentiable, then g' satisfies the conclusion of the Intermediate Value Theorem, even though g' may not be continuous.

4. Let S be the space of all infinite sequences of 0's and 1's, that is, $S = \{(s_1, s_2, \ldots) : s_i \in \{0,1\}\}$. Define a metric d on S by setting

$$d((s_1, s_2, \ldots), (t_1, t_2, \ldots)) = \begin{cases} \frac{1}{n} & \text{if } s_n \neq t_n \text{ and } s_i = t_i \text{ for } 1 \leq i < n, \\ 0 & \text{if } s_i = t_i \text{ for all } i. \end{cases}$$

(That is, if two sequences first differ at the tenth spot, then the distance between them is $1/10$.)

(a) Show that d really is a metric.

(b) Is the map $\sigma : S \to S$ given by $\sigma(s_1, s_2, \ldots) = (s_2, s_3, \ldots)$ continuous?

(c) Are the following subsets of S open? Closed?
  (i) The set A of all sequences that contain exactly two 1's.
  (ii) The set B of all sequences that begin with a 0.

5. Define a sequence of functions $f_1, f_2, \ldots : [0,1] \to \mathbb{R}$ by $f_n(x) = \dfrac{nx^n}{1+nx^n}$. Discuss the convergence of $\{f_n\}$, $\{f_n'\}$, and $\{\int_0^1 f_n(x)\,dx\}$ as $n \to \infty$. If the convergence isn't nice, explain what additional hypotheses would make it nice.

Complex analysis

6. Prove or disprove the following statement: Let $f(z)$ be an entire function. If f is bounded, then f is constant.

7. Prove or disprove the following statement: If $\{f_n\}_{n=1}^{\infty}$ is a sequence of entire functions that converges uniformly to a function f in every compact subset of $\mathbb{C}$, then f is entire.

8. Let f be a nonwhere-vanishing holomorphic function on a simply connected region Ω. Show that f has a square root, that is, show that there exists a holomorphic function g on Ω such that $f(z) = (g(z))^2$ for all $z \in \Omega$.

9. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. We say that a point z_0 is a *fixed point* for f if $f(z_0) = z_0$; it's an *attracting* fixed point if there exists a disk D centered at z_0 such that $f(D) \subset D$ and the sequence $z, f(z), f(f(z)), f(f(f(z))), \ldots$ approaches z_0 for every z in D. Prove the following theorem: A fixed point z_0 is attracting if and only if $|f'(z_0)| < 1$.

10. (a) Compute the integral $\displaystyle\int_0^{\infty} \frac{\cos x}{x^2+1}\, dx$.

(b) Find all solutions to the equation $z = 3e^z$ in the closed unit disk.