Swarthmore College
Department of Mathematics and Statistics
2017 Honors Examination in Algebra

This exam consists of 10 problems. You are not expected to solve them all, but rather should work on the ones you find interesting and approachable. The exam is divided into three sections - please attempt to solve at least one, and ideally at least two, problems from each section.

I am interested in seeing how you approach the various problems, so please turn in your solutions to a problem even if you can only make progress on some of the individual parts. If you find it useful, you may assume an earlier part of problem when working on the later parts even if you haven't been able to solve it.

Section I

(1) Let x, y be 3-cycles in S_5, not equal or inverse to each other.

(a) Prove that if x and y fix a common point then they generate a copy of A_4 in S_5.

(b) Prove that if x and y do not share a common fixed point, then they generate A_5.

(2) Let G be a group of size 105. Let P be a Sylow 5-subgroup and let Q be a Sylow 7-subgroup.

(a) Prove that at least one of P or Q must be normal in G.

(b) Prove that PQ is a normal subgroup of G of size 35. (Hint: Show that any subgroup of size 35 in G must be normal.)

(c) Prove that both P and Q are normal in G.

(3) Let $A = \{(a, b) : 1 \leq a, b \leq 4\}$ and let S_4 operate on A by operating on the entries in each ordered pair. That is, for $(a, b) \in A$ and $\sigma \in S_4$ we set

$$\sigma * (a, b) = (\sigma(a), \sigma(b)).$$

Let V_A be the complex vector space with basis $\{e_x : x \in A\}$ and let $\rho_A \colon S_4 \to \mathrm{GL}(V_A)$ be the associated permutation representation of S_4. Let χ_A be the character of ρ_A.

You may use of the character table for S_4 for this problem, here it is:

	e	(1 2)	(1 2)(3 4)	(1 2 3)	(1 2 3 4)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

(a) Compute the values of the character χ_A. You can present your answer by filling in a chart such as:

	e	(1 2)	(1 2)(3 4)	(1 2 3)	(1 2 3 4)
χ_A					

(b) How does ρ_A decompose as a sum of irreducible representations? That is, let ρ_i be the representation corresponding the character χ_i as given in the table above and find integers $d_1, \ldots, d_5$ so that ρ_A is isomorphic to $d_1\rho_1 \oplus \cdots \oplus d_5\rho_5$.

(c) Find the d_1-dimensional subspace of V_A on which S_4 acts trivially.

Section II

(4) Let V be the set of infinitely differentiable functions from $\mathbf{R}$ to $\mathbf{C}$ that are periodic with period 1. That is

$$V = \{f \colon \mathbf{R} \to \mathbf{C} : f \text{ is infinitely differentiable and } f(x+1) = f(x) \text{ for all } x\}.$$

Note that V is a complex vector space. For $f, g \in V$ we define

$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}dx.$$

(a) Prove that $\langle\ ,\ \rangle$ is a Hermitian form on V.

(b) Prove that the set $\{f_n(x) = e^{inx} : n \in \mathbf{N}\}$ is an orthonormal set with respect to $\langle\ ,\ \rangle$.

(c) Let $T \colon V \to V$ be given by $T(f) = f'$ and let $S \colon V \to V$ be given by $S(f) = -f'$. Here f' denotes the derivative of f. Prove that S is the adjoint of T with respect to $\langle\ ,\ \rangle$.

(5) (a) Show that the elements 2,3, $1 \pm \sqrt{-5}$ are irreducible in $\mathbf{Z}[\sqrt{-5}]$.

(b) Show that $\mathbf{Z}[\sqrt{-5}]$ is not a UFD. (Hint: why part (a)?)

(c) Explain how to prove that $\mathbf{Z}[\sqrt{-2}]$ *is* a UFD. You do not have to give a complete proof, but you should indicate the major steps along the chain of reasoning that allows you to draw this conclusion.

(6) (a) Show that $x^5 + 9x^4 - 3x^2 + 3$ and $x^4 + 3x^3 + 6x^2 + 2x + 5$ are both irreducible in $\mathbf{Z}[x]$.

(b) Give an example of a maximal ideal of $\mathbf{Z}[x]$. Given an example of an ideal of $\mathbf{Z}[x]$ which is prime but not maximal. Given an example of an ideal of $\mathbf{Z}[x]$ which is neither maximal nor principal.

(c) Every ideal of $\mathbf{Z}[x]$ is finitely generated (you don't need to prove that). However, there are ideals that need arbitrarily large generating sets. Find an ideal of $\mathbf{Z}[x]$ that requires 3 generators. That is, $I = (f, g, h)$ for some polynomials $f, g, h \in \mathbf{Z}[x]$ but I is not equal to (a, b) for any $a, b \in \mathbf{Z}[x]$. Can you find an ideal that requires 1000 generators?

(7) (a) How many isomorphism classes are there of abelian groups of size 72? Give a representative from each class.

(b) Let

$$A = \begin{bmatrix} 0 & 12 & -6 \\ 2 & 6 & -6 \\ 4 & 18 & -18 \end{bmatrix}.$$

Find invertible integral matrices P and Q so that QAP is diagonal.

(c) Let R be the subgroup of $\mathbf{Z}^3$ generated by the elements $(0, 12, -6), (2, 6, -6), (4, 18, -18)$. The group $G = \mathbf{Z}^3/R$ has size 72. Which of the groups you named in the first part is isomorphic to G?

Section III

(8) Let $K = \mathbf{Q}(\sqrt[3]{5})$.

(a) Give a basis for K as a $\mathbf{Q}$ vector space.

(b) Let $T\colon K \to K$ be given by $T(x) = (1 + \sqrt[3]{5} + 2\sqrt[3]{25})x$. What is the matrix for T with respect to the basis you gave in the previous part?

(c) What is the minimal polynomial of $1 + \sqrt[3]{5} + 2\sqrt[3]{25}$ over $\mathbf{Q}$?

(9) (a) Let E/F be a field extension of finite dimension. Let α be in E. Prove that α is algebraic over F.

(b) Suppose $E/\mathbf{Q}$ is Galois and has Galois group A_4. Which numbers occur as the degrees of minimal polynomials of elements of E?

(c) Suppose $f(x) \in \mathbf{Q}[x]$ is an irreducible polynomial of degree 4 with Galois group A_4, and suppose $\alpha \in \mathbf{R}$ is a root of $f(x)$. Is α a constructible number?

(10) Let K be the splitting field of $x^6 - 2$ over $\mathbf{Q}$ and let G be the Galois group of K over $\mathbf{Q}$.

(a) Determine $[K : \mathbf{Q}]$.

(b) Find two quadratic extensions of $\mathbf{Q}$ contained in K.

(c) Use the previous parts to conclude that the 3-Sylow subgroup of G is normal.

(d) Is the 2-Sylow subgroup of G normal?