Swarthmore College
Honors Examination in Topology, 2016

Instructions. Solve at least 6 problems, at least one from each section, and ideally at least two from each section. (There are three sections.)

(I) Point-set topology

1. (a) Give an example of a space which is path connected but not locally path connected.

   (b) Assume that X is locally path connected. Prove that if U is an open subset of X, then U is connected if and only if U is path connected.

2. A space X is *limit point compact* if every infinite subset of X has a limit point. A space X is *countably compact* if every countable cover has a finite subcover.

   (a) Prove that the continuous image of a countably compact space is countably compact.

   (b) If X is countably compact, prove that X is limit point compact.

   (c) If X is limit point compact and Hausdorff, prove that X is countably compact.

3. A topological space is *normal* if for every pair of disjoint closed sets, there are disjoint open sets containing them. Prove that every compact Hausdorff space is normal.

(II) π_1, homotopy

4. Suppose that X, Y, and Z are path connected and locally path connected, and suppose we have continuous functions $p : X \to Y$ and $q : Y \to Z$. You may use the fact that if p and $q \circ p$ are covering maps, so is q.

   (a) Assume that $(q \circ p)^{-1}(z)$ is finite for each $z \in Z$. Prove that if p and q are covering maps, so is $q \circ p$.

   (b) Assume that Z has a universal cover. Prove that if p and q are covering maps, so is $q \circ p$.

   (c) Prove that if q and $q \circ p$ are covering maps, so is p.

5. Let G be a topological group with identity e. Let $*$ denote path multiplication and the corresponding induced multiplication in $\pi_1(G, e)$. Write $\cdot$ for the group operation in G, and use it to define an operation $\cdot$ on $\pi_1(G, e)$: for loops α and β, define $(\alpha \cdot \beta)(t) = \alpha(t) \cdot \beta(t)$. Prove that $[\alpha] * [\beta] = [\alpha] \cdot [\beta]$ for all $[\alpha], [\beta] \in \pi_1(G, e)$, and prove that $\pi_1(G, e)$ is abelian.

6. The hypotheses of the Seifert-van Kampen theorem are: $X = U \cup V$ where U and V are open in X; U, V, and $U \cap V$ are path connected; and $U \cap V$ is nonempty, in particular containing the base point for all of the relevant π_1 calculations. What happens when you drop some of these hypotheses?

   (a) Give a counterexample to the theorem in which U is not open in X (but the rest of the hypotheses hold).

   (b) Give a counterexample in which $U \cap V$ is not path connected (but the rest hold).

   (c) Is there a counterexample in which U is not path connected (but the rest hold)?

(III) Homology

7. (a) If K and L are one-dimensional simplicial complexes, compute the Euler characteristic $\chi(K \times L)$ in terms of $\chi(K)$ and $\chi(L)$.

   (b) If M and N are compact surfaces without boundary, compute the Euler characteristic $\chi(M \# N)$ in terms of $\chi(M)$ and $\chi(N)$.

   (c) What are the Euler characteristics of the circle, the sphere, and the real projective plane?

   (d) Use parts (a)–(c) to compute the Euler characteristic of every compact connected surface without boundary.

8. One version of the Snake Lemma says that, given a commutative diagram

$$\begin{array}{ccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & & \end{array}$$

   of abelian groups where the rows are exact, there is an exact sequence

$$\ker a \to \ker b \to \ker c \xrightarrow{d} \operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} c,$$

   where $\operatorname{coker} a$ is the *cokernel of* a, defined to be $A' / \operatorname{im} a$, the codomain mod the image.

   Define the map d and prove exactness at $\ker c$ and at $\operatorname{coker} a$.

9. Suppose that G is a compact, path connected, triangulable topological group with more than one element. Prove that its Euler characteristic is zero. (Hint: fix $g \in G$ and consider the map $x \mapsto gx$.)