Swarthmore College Honors Examination in Topology, 2016

Instructions. Solve at least 6 problems, at least one from each section, and ideally at least two from each section. (There are three sections.)

(I) Point-set topology —

- 1. (a) Give an example of a space which is path connected but not locally path connected.
 - (b) Assume that X is locally path connected. Prove that if U is an open subset of X, then U is connected if and only if U is path connected.
- 2. A space X is *limit point compact* if every infinite subset of X has a limit point. A space X is *countably compact* if every countable cover has a finite subcover.
 - (a) Prove that the continuous image of a countably compact space is countably compact.
 - (b) If X is countably compact, prove that X is limit point compact.
 - (c) If X is limit point compact and Hausdorff, prove that X is countably compact.
- 3. A topological space is *normal* if for every pair of disjoint closed sets, there are disjoint open sets containing them. Prove that every compact Hausdorff space is normal.

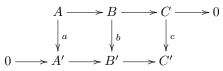
(II) π_1 , homotopy —

- 4. Suppose that X, Y, and Z are path connected and locally path connected, and suppose we have continuous functions $p: X \to Y$ and $q: Y \to Z$. You may use the fact that if p and $q \circ p$ are covering maps, so is q.
 - (a) Assume that $(q \circ p)^{-1}(z)$ is finite for each $z \in Z$. Prove that if p and q are covering maps, so is $q \circ p$.
 - (b) Assume that Z has a universal cover. Prove that if p and q are covering maps, so is $q \circ p$.
 - (c) Prove that if q and $q \circ p$ are covering maps, so is p.
- 5. Let G be a topological group with identity e. Let * denote path multiplication and the corresponding induced multiplication in $\pi_1(G, e)$. Write \cdot for the group operation in G, and use it to define an operation \cdot on $\pi_1(G, e)$: for loops α and β , define $(\alpha \cdot \beta)(t) = \alpha(t) \cdot \beta(t)$. Prove that $[\alpha] * [\beta] = [\alpha] \cdot [\beta]$ for all $[\alpha], [\beta] \in \pi_1(G, e)$, and prove that $\pi_1(G, e)$ is abelian.

- 6. The hypotheses of the Seifert-van Kampen theorem are: $X = U \cup V$ where U and V are open in X; U, V, and $U \cap V$ are path connected; and $U \cap V$ is nonempty, in particular containing the base point for all of the relevant π_1 calculations. What happens when you drop some of these hypotheses?
 - (a) Give a counterexample to the theorem in which U is not open in X (but the rest of the hypotheses hold).
 - (b) Give a counterexample in which $U \cap V$ is not path connected (but the rest hold).
 - (c) Is there a counterexample in which U is not path connected (but the rest hold)?

(III) Homology —

- 7. (a) If K and L are one-dimensional simplicial complexes, compute the Euler characteristic $\chi(K \times L)$ in terms of $\chi(K)$ and $\chi(L)$.
 - (b) If M and N are compact surfaces without boundary, compute the Euler characteristic $\chi(M \# N)$ in terms of $\chi(M)$ and $\chi(N)$.
 - (c) What are the Euler characteristics of the circle, the sphere, and the real projective plane?
 - (d) Use parts (a)–(c) to compute the Euler characteristic of every compact connected surface without boundary.
- 8. One version of the Snake Lemma says that, given a commutative diagram



of abelian groups where the rows are exact, there is an exact sequence

 $\ker a \to \ker b \to \ker c \xrightarrow{d} \operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} c,$

where coker a is the *cokernel of* a, defined to be $A'/\operatorname{im} a$, the codomain mod the image.

Define the map d and prove exactness at ker c and at coker a.

9. Suppose that G is a compact, path connected, triangulable topological group with more than one element. Prove that its Euler characteristic is zero. (Hint: fix $g \in G$ and consider the map $x \mapsto gx$.)