SWARTHMORE COLLEGE HONORS EXAM 2016
REAL ANALYSIS

Instructions: Submit responses to as many of the following questions as you can. Even if you are not able to completely solve a particular problem, submit your most promising partial progress and briefly indicate how you expect the rest of the solution to proceed. You may refer to major results (e.g., named theorems) without proof unless strictly forbidden by a problem, but please be clear about the hypotheses and conclusions.

1. Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers. Let S_N be the Nth partial sum of the infinite series $\sum_{n=1}^{\infty} a_n$, i.e., $S_N := a_1 + \cdots + a_N$. We say that the series $\sum_{n=1}^{\infty} a_n$ is *Cesàro summable with value L* when $\frac{1}{N}(S_1 + \cdots + S_N) \to L$ as $N \to \infty$.
   (a) Prove that every convergent series is Cesàro summable with value equal to the sum of the series.
   (b) Construct (with proof) a divergent series which is Cesàro summable.
2. (a) Suppose K is a nonempty compact subset of a metric space X with metric d. Show that if K is not connected, then there exist nonempty closed sets K_1 and K_2 and a real number $\epsilon > 0$ such that $K = K_1 \cup K_2$ and $d(x_1, x_2) > \epsilon$ for all $x_1 \in K_1$ and all $x_2 \in K_2$.
   (b) Give an example (with proof) of a metric space X and a closed (non-compact) set $K \subset X$ which is not connected and fails to satisfy the conclusion of part (a).
3. Let $F : \mathbb{R}^n \to \mathbb{R}$ be the function
$$F(x_1, \ldots, x_n) := \sup_{t \in [0,1]} \left| x_1 + x_2 t + \cdots + x_n t^{n-1} \right|.$$
   Show that F is continuous and then show that $F(x_1, \ldots, x_n) = 0$ if and only if $(x_1, \ldots, x_n) = (0, \ldots, 0)$.
4. Suppose f is a continuous, real-valued function on the closed interval $[0, 1]$ which is differentiable at every point in the open interval $(0, 1)$ and has $f(0) = f(1) = 0$. Show that for any $x_0 \notin [0, 1]$, there exists $c \in (0, 1)$ such that the tangent line $y = f'(c)(x - c) + f(c)$ passes through the point $(x_0, 0)$.
5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by
$$f(x, y) := \begin{cases} \sin \frac{1}{x^2+y^2-1} & x^2 + y^2 \neq 1, \\ 0 & x^2 + y^2 = 1 \end{cases}.$$
   Prove from the definition that f is Riemann integrable on the square $[-2, 2]^2$.

6. Let $T \subset \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ be the set of all triples (x_1, x_2, x_3) of points in $\mathbb{R}^3$ which collectively form the vertices of an equilateral triangle, i.e.,

$$T := \left\{ (x_1, x_2, x_3) \in (\mathbb{R}^3)^3 \; : \; ||x_1 - x_2|| = ||x_2 - x_3|| = ||x_3 - x_1|| > 0 \right\}.$$

(Here $|| \cdot ||$ denotes the usual Euclidean distance.) Show that T is a manifold and compute its dimension. Then find (with proof) a basis for all vectors tangent to T at (x_1, x_2, x_3) when

$$x_1 = (1, 0, 0), \;\; x_2 = (0, 1, 0), \;\; x_3 = (0, 0, 1).$$

7. Let $B := \{(x, y) \in \mathbb{R}^2 \; : \; x^2 + y^2 \leq 1\}$, and suppose that $f : B \to \mathbb{R}$ is C^2 on the interior of B and continuous on all of B.

   (a) Show that there exists $\delta > 0$ such that, for all $t \in (-\delta, \delta)$, the function

   $$1 - x^2 - y^2 + t f(x, y) \tag{$\dagger$}$$

   has a unique critical point in the interior of B. For t in this same interval, show that the maximum of ($\dagger$) is attained at the critical point and the minimum is attained on the boundary.

   (b) For each $t \in (-\delta, \delta)$, let $\gamma(t)$ equal the location of the interior critical point of ($\dagger$). Prove that there exists $\delta' \in (0, \delta]$ for which γ is C^1 on $(-\delta', \delta')$.

8. (a) Suppose ω is a smooth differential k-form on some n-dimensional manifold with $k \leq n$. If k is odd, prove that $\omega \wedge \omega = 0$.

   (b) Construct (with proof) an example of a smooth differential 2-form ω on $\mathbb{R}^4$ such that $\omega \wedge \omega$ does not vanish at any point.

9. Suppose $\mathbb{S}^2$ denotes the unit sphere in $\mathbb{R}^3$. For each integer $k \geq 1$, let

$$M_k := \left\{ (x, y, z) \in \mathbb{R}^3 \; : \; x^{2k} + y^2 + z^2 = 1 \right\}$$

and let $\pi_k : M_k \to \mathbb{S}^2$ be given by $\pi_k(x, y, z) := (x^k, y, z)$. For each k, give M_k the standard orientation and compute (with full justification)

$$\int_{M_k} \pi_k^*(x \; dy \wedge dz - y \; dx \wedge dz + z \; dx \wedge dy).$$