Swarthmore College
Department of Mathematics and Statistics

Honors Examination in Geometry 2017

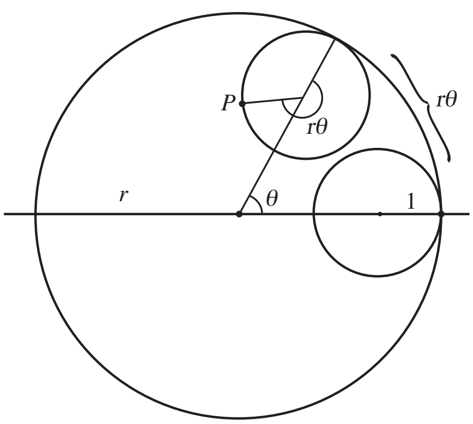
Instructions: Do as many of the following 12 problems as thoroughly as you can in the time you have. Try to include at least one problem from each of the four parts of the exam. You may use without proof the basic theorems that you have learned, but be sure to state them carefully. You may also use a result from one problem in solving another, even if you didn't solve the first problem. If you have lots of time left over, feel free to solve other problems.

Notation

The following conventions for notation have been followed:

$\mathbb{R}$ = the set (group, field) of real numbers.
$\mathbb{C}$ = the set (group, field) of complex numbers.
S^{n-1} = the unit sphere in $\mathbb{R}^n$.

Curves



1. The *hypocycloid* is the curve made when a point on a circle is rotated along the inside of a circle without slipping. As pictured, the point P lies on the hypocycloid generated by rolling a circle of radius one counterclockwise inside a circle of radius r. Parametrize this hypocycloid and use your parametrization to compute the length of the curve after the angle θ goes from 0 to 2π. For what values of r is the hypocycloid a closed curve?

2. A curve in the plane is given by polar coordinates of the form $r = r(\theta)$. What is the expression in terms of r for the signed plane curvature of such a curve? Consider the spirals: Archimedean, $r = a\theta$; logarithmic, $r = ae^{k\theta}$, and asymptotic, $r = 1 - e^{-k\theta}$. What are their curvatures? How do the curvatures behave as θ goes to infinity? Does this make sense?

3. Suppose that $\gamma(s)$ is a unit-speed space curve with the following property: there are constants a, b, and c for which the linear combination $aT(s)+bN(s)+cB(s) = \mathbf{v}$ gives a constant vector, where $\{T(s), N(s), B(s)\}$ is the Frenet frame along γ. Suppose the curvature of γ does not vanish. Prove that the ratio between the curvature κ and the torsion τ of γ is constant. If you have time, show that a circular helix satisfies this condition.

Surfaces, curvature

4. A particular example of a surface in $\mathbb{R}^3$ is the graph of a function $z = f(x, y)$. Show how the normal to this surface is a gradient. Determine the associated coefficients of the first and second fundamental form. In the case of

$$f(x, y) = ax^2 + bxy + cy^2 + dx + ey + h,$$

show how the discriminant of f, $\Delta = b^2 - 4ac$, determines the sign of the curvature of the surface $z = f(x, y)$.

5. The *monkey saddle* is the graph of the function $f(u, v) = u^3 - 3v^2u$. This surface is interesting at the origin. Show that a plane containing the z-axis meets the surface in a curve that is a cubic like $y = x^3$ with inflection point at the origin. Using the coordinate chart $X(u, v) = (u, v, u^3 - 3v^2u)$, compute the Gaussian curvature of the surface at the origin.

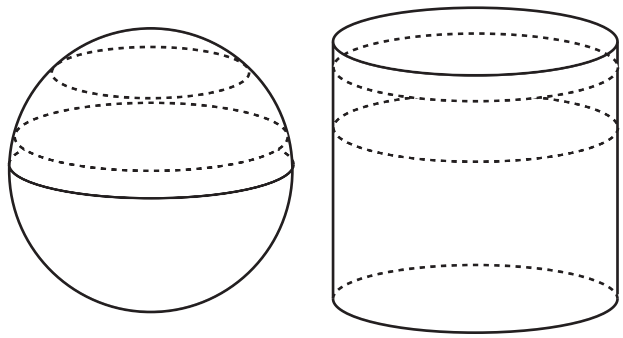
6. The open cone F_1 given by the equation $x^2+y^2 = z^2$ $(z > 0)$ and the cylinder F_2 over the unit circle $\{(x, y, z) \mid x^2+y^2 = 1, z > 0\}$ are both surfaces of zero curvature. Give a geometric reason why. Consider the mapping $\phi\colon (r\cos\theta, r\sin\theta, r) \mapsto (\cos\theta, \sin\theta, r)$ taking a point in the cone to a point on the cylinder. At the point $P = (1, 0, 1)$ in the cone, what is the linear mapping $d\phi_P\colon T_PF_1 \to T_PF_2$ on tangent planes? Is $d\phi_P$ an isometry of vector spaces?

Geometry on surfaces

7. Prove in two different ways that sum of the interior angles of a triangle in the Euclidean plane is π, two right angles.

8. Clairaut's relation on a surface of revolution states that a geodesic $\gamma(s)$ on a surface of revolution that is not a parallel (one of circles perpendicular to the axis of rotation) satisfies the relation $\rho\sin\psi =$ a constant where ρ is the distance from the point $\gamma(s)$ to the axis and ψ is the angle $\gamma'(s)$ makes with the meridian through $\gamma(s)$. Let's explore this relation on a cone, generated by a line through the origin $y = mx$. Given a geodesic $\gamma(s)$ and an initial point $\gamma(s_0)$ with $\rho(s_0)$ and $\sin\psi(s_0)$ obtained from $\gamma(s_0)$ and $\gamma'(s_0)$, how far can the geodesic go along the curve before it has to turn around and return?

9. Using differential geometry, prove Archimedes's Theorem that the projection of the sphere onto a circular cylinder of the same radius as the sphere and of height the diameter of the sphere preserves area. The projection is given by taking a plane perpendicular to the axis of the cylinder and mapping the circles of intersection to one another by central projection.



Geometry more generally

10. The unit sphere S^2 in $\mathbb{R}^3$ has the longitude-latitude parametrization, $\mathbf{x}\colon (-\pi.\pi)\times(-\pi/2,\pi/2) \to \mathbb{R}^3$ given by $\mathbf{x}(\theta,\phi) = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$. If the metric is inherited from space $\mathbb{R}^3$, then there is a unique Riemannian connection on S^2 compatible with the metric. Determine the coefficients of the metric. Determine the vector fields $\nabla_{\frac{\partial}{\partial\theta}}\dfrac{\partial}{\partial\theta}$, $\nabla_{\frac{\partial}{\partial\phi}}\dfrac{\partial}{\partial\theta}$, and $\nabla_{\frac{\partial}{\partial\phi}}\dfrac{\partial}{\partial\phi}$. Recall that $\nabla_{\frac{\partial}{\partial u_i}}\dfrac{\partial}{\partial u_j} = \sum_k \Gamma_{ij}^k \dfrac{\partial}{\partial u_k}$ and $\Gamma_{ij}^m = \dfrac{1}{2}\displaystyle\sum_k \left\{\frac{\partial}{\partial u_i}g_{jk} + \frac{\partial}{\partial u_j}g_{ki} - \frac{\partial}{\partial u_k}g_{ij}\right\} g^{km}$.

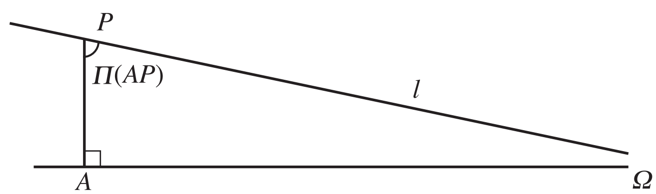
11. The Poincaré half-plane is a surface with $\mathbb{H} = \{(x,y)\in\mathbb{R}^2 \mid y>0\}$ as chart and Riemannian metric given by

$$ds^2 = \frac{dx^2+dy^2}{y^2}.$$

Consider the family of curves $\gamma_s(t) = (s\cos t, s\sin t)$ where $0<t<\pi$. For a fixed s_0 compute the arc length of $\gamma_{s_0}(t)$ for the values $\dfrac{\pi}{2} \le t \le \dfrac{3\pi}{2}$. What does your answer tell you about the geometry of the hyperbolic plane?

12. The Poincaré half-plane is a model of non-Euclidean geometry, which means that the theorems proved by Lobachevsky and Bolyai should be able to be proved for figures in the model. An idea in the non-Euclidean plane is the *angle of parallelism* $\Pi(AP)$ of a line segment AP: it is the angle made with a perpendicular AP to a line $A\Omega$ by the line through P that is the "first" parallel, that is, the line l does not meet

$A\Omega$, but any other line through P making a smaller angle with AP would meet $A\Omega$.



In the Poincaré half-plane, lines are vertical lines and the semi-circles with center on the real axis. Let $A = (0, 1)$ and $P = (0, y)$ lie on the vertical line through the origin. Let $A\Omega$ denote the upper unit circle centered at the origin. Letting Ω lie on the real line, the first parallel through P to $A\Omega$ passes through Ω as well (since Ω is not in $\mathbb{H}$). The Lobachevsky-Bolyai Theorem states $\tan\left(\frac{\Pi(AP)}{2}\right) = e^{-|AP|}$. Using the conformality of the Poincaré half-plane and the Euclidean geometry of the upper half-plane, prove the Lobachevsky-Bolyai theorem. (Remember, you need to know $|AP|$ in $\mathbb{H}$.)