SWARTHMORE COLLEGE HONORS EXAM 2016
COMPLEX ANALYSIS

Instructions: Submit responses to as many of the following questions as you can. Even if you are not able to completely solve a particular problem, submit your most promising partial progress and briefly indicate how you expect the rest of the solution to proceed. You may refer to major results (e.g., named theorems) without proof unless strictly forbidden by a problem, but please be clear about the hypotheses and conclusions.

1. Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers. Let S_N be the Nth partial sum of the infinite series $\sum_{n=1}^{\infty} a_n$, i.e., $S_N := a_1 + \cdots + a_N$. We say that the series $\sum_{n=1}^{\infty} a_n$ is *Cesàro summable with value L* when $\frac{1}{N}(S_1 + \cdots + S_N) \to L$ as $N \to \infty$.

   (a) Prove that every convergent series is Cesàro summable with value equal to the sum of the series.

   (b) Construct (with proof) a divergent series which is Cesàro summable.

2. (a) Suppose K is a nonempty compact subset of a metric space X with metric d. Show that if K is not connected, then there exist nonempty closed sets K_1 and K_2 and a real number $\epsilon > 0$ such that $K = K_1 \cup K_2$ and $d(x_1, x_2) > \epsilon$ for all $x_1 \in K_1$ and all $x_2 \in K_2$.

   (b) Give an example (with proof) of a metric space X and a closed (non-compact) set $K \subset X$ which is not connected and fails to satisfy the conclusion of part (a).

3. Suppose f is a continuous, real-valued function on the closed interval $[0, 1]$ which is differentiable at every point in the open interval $(0, 1)$ and has $f(0) = f(1) = 0$. Show that for any $x_0 \notin [0, 1]$, there exists $c \in (0, 1)$ such that the tangent line $y = f'(c)(x - c) + f(c)$ passes through the point $(x_0, 0)$.

4. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of nonnegative Riemann integrable functions on $[0, 1]$. Suppose also that $\forall x \in [0, 1]$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ is strictly decreasing and that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0.$$

   (a) Show that $f(x) := \lim_{n\to\infty} f_n(x)$ is Riemann integrable on $[0, 1]$ and that

$$\int_0^1 f(x)dx = 0.$$

   (b) Show that there must be a dense set $E \subset [0, 1]$ such that for all $x \in E$,

$$0 = f(x) = \lim_{y\to x} f(y).$$

5. (a) Identify all zeros, poles, removable singularities, and/or essential singularities of

$$f(z) := \frac{1}{z} \sin\left(\frac{\pi z}{1-z}\right).$$

For any zeros or poles, give the orders.

(b) Prove that the function f above has a Laurent series expansion of the form

$$f(z) = \sum_{n=0}^{\infty} \frac{c_n}{(z-1)^n}$$

which converges on $\mathbb{C} \setminus \{1\}$. Compute c_n for $n \leq 3$.

6. (a) Suppose $\omega \in \mathbb{C}$ satisfies $|\omega| > 1$. Show that there exists a unique entire function f such that $f(0) = 1$ and

$$f(\omega z) = (1-z) f(z) \text{ for all } z \in \mathbb{C}. \qquad (\dagger)$$

Then show that $f(z) = 0$ if and only if $z = \omega^n$ for some positive integer n.

(b) If $|\omega| < 1$, show that there is a unique holomorphic f on the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ satisfying ($\dagger$) and the constraint $f(0) = 1$. Show that f has no zeros in the unit disk and cannot be extended to an entire function.

7. Use complex techniques to evaluate the improper integral

$$\int_{-\infty}^{\infty} e^{iax} \frac{\sin x}{x} dx$$

for all real numbers $a \neq \pm 1$.

8. (a) Suppose f and g are holomorphic functions on $D_2 := \{z \in \mathbb{C} : |z| < 2\}$ and that the real parts of f and g are equal at all points on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Show that there is a real constant c such that $f(z) = g(z) + ic$ for all $z \in D_2$.

(b) What, if anything, changes if the real parts of f and g are known to be equal on a line through the origin instead of the unit circle?

9. Show that the punctured unit disk $\{z \in \mathbb{C} : 0 < |z| < 1\}$ is not conformally equivalent to the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$, i.e., show that there is no bijection φ of these two sets for which φ and φ^{-1} are both holomorphic.