Swarthmore College
Department of Mathematics and Statistics
Honors Examination: Algebra
Spring 2016

Instructions: This exam contains nine problems. Try to solve *six* problems as completely as possible. Do not be concerned if some portions of the exam are unfamiliar; you have a number of choices so that you do not need to answer every question. Once you are satisfied with your responses to six problems, make a second pass through the exam and complete as many parts of the remaining problems as possible. I am interested in your thoughts on a problem even if you do not completely solve it. In particular, submit your solution even if you cannot do all the parts of a problem. When there are multiple parts, you are permitted to address a later part without solving all the earlier ones.

1. Let $G = C_2 \times C_2 = \langle a, b \mid abab = a^2 = b^2 = 1\rangle$, and define $\rho : G \to \mathrm{GL}_2(\mathbb{C})$ to be the representation given by

$$\begin{aligned} a &\mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \\ b &\mapsto \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}. \end{aligned}$$

(a) Is ρ is an irreducible representation over $\mathbb{C}$? Justify your answer.
(b) Would your answer to part (a) change if you were working over $\mathbb{R}$ instead of $\mathbb{C}$? (You do not need to have studied representations over $\mathbb{R}$ previously to work on this problem.)
(c) Produce all the irreducible representations of G over $\mathbb{C}$ of dimension 1.
(d) Would your answer in part (c) change if you were working over $\mathbb{R}$ instead of $\mathbb{C}$?

2. Let H, K, and N be subgroups of a group G such that $N \trianglelefteq G$ and $K \trianglelefteq H$.

(a) Prove that every subgroup of H/K is of the form M/K where $K \subseteq M \subseteq H$.
(b) Prove that there exists a normal subgroup of H/K that is isomorphic to $(H \cap N)/(K \cap N)$.
(c) Prove that there exists a quotient group of H/K that is isomorphic to HN/KN.

3. Let R be a commutative ring with multiplicative identity that is not a field. Here we define ideals to be proper. Prove that the following conditions are equivalent.

(a) If $r, s \in R$ are non-invertible, then $r + s$ is non-invertible.
(b) The non-invertible elements of R form an ideal.
(c) The ring R possesses a unique maximal ideal. [An ideal I of R is said to be *maximal* if there is no ideal J of R that properly contains I.]

4. Consider the ring of integers $\mathbb{Z}$, and let m and n be positive integers.

(a) For what values of m is $\mathbb{Z}/m\mathbb{Z}$ an integral domain?
(b) For what values of m is $\mathbb{Z}/m\mathbb{Z}$ a field?
(c) For what values of m is $\mathbb{Z}/m\mathbb{Z}$ a Euclidean domain?
(d) If m and n are distinct, is it possible for $m\mathbb{Z}$ and $n\mathbb{Z}$ to be isomorphic as rings? If so, what are the conditions on m and n for this to occur?

5. Consider the polynomials $f(x) = x^4 - 2$ and $g(x) = x^4 + 2$ over $\mathbb{Q}$.

(a) Determine the splitting field F of f over $\mathbb{Q}$.
(b) What is the degree of F over $\mathbb{Q}$?
(c) Find a basis for F over $\mathbb{Q}$.
(d) Show that F is the splitting field of g over $\mathbb{Q}$.
(e) Can you produce infinitely many irreducible, monic polynomials $f_1, f_2, \ldots$ such that the splitting fields of these polynomials are identical to F?

6. Let p be a prime integer and $n > 1$ be a positive integer. Let $\mathbb{F}_p \subset \mathbb{F}_{p^n}$ be an extension of finite fields. Define $\Phi : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ to be the automorphism given by

$$\Phi(\alpha) = \alpha^p.$$

(a) Note that $\mathbb{F}_{p^n}$ can be viewed as a vector space over $\mathbb{F}_p$. Show that Φ is an $\mathbb{F}_p$-linear map.
(b) Suppose $m_\Phi(X) = \sum c_i \Phi(X)^i \in \mathbb{F}_p[X]$ is the minimal polynomial of Φ. This means that $m_\Phi(\alpha) = 0$ for all $\alpha \in \mathbb{F}_{p^n}$. Show that $m_\Phi(X)$ has degree n.

7. Consider the alternating group A_5, which consists of even permutations on a set of 5 elements. Compute the number of p-Sylow subgroups of A_5.

8. For elements g, h of a group G, define the commutator of g and h as

$$[g, h] = ghg^{-1}h^{-1}.$$

In general, the set of commutators

$$S_G = \{[g, h] \mid g, h \in G\}$$

need not be a group. We call the smallest subgroup of G containing S_G the commutator subgroup of G, which we denote G'. We define the center of G to be

$$Z(G) = \{h \in G \mid gh = hg \text{ for all } g \in G\}.$$

For $x \in G$, define the set of conjugates of x to be $C_x = \{gxg^{-1} \mid g \in G\}$, and define $B_x = C_x C_{x^{-1}} = \{ab \mid a \in C_x, b \in C_{x^{-1}}\}$.

(a) Prove that $S_G = \bigcup_{x \in G} B_x$.
(b) Show that if $y \in B_x$, then $C_y \subseteq B_x$.
(c) Show that if $y \in B_x$, then $y^{-1} \in B_x$.
(d) Suppose $y = cx'$ where $c \in Z(G)$ and $x' \in C_x$. Prove that $B_x = B_y$.
(e) Suppose for some $x \in G$, B_x is a subgroup of G that contains S_G. Prove that $S_G = B_x = G'$.

9. For each of the following rings, determine whether every ideal is principal.

(b) $\mathbb{Z}[X, Y]$
(a) $\mathbb{Z}[X] \oplus \mathbb{Z}[Y]$
(c) $(\mathbb{Z}/2\mathbb{Z})[X]$
(d) $(\mathbb{Z}/4\mathbb{Z})[X]$
(e) $(\mathbb{Z}/6\mathbb{Z})[X]$