ELSEVIER

Topology and its Applications 98 (1999) 203–210

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

Recurrent sequences and IP sets

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Received 3 September 1997; received in revised form 9 January 1998

Abstract

A strong connection exists between combinatorial properties and dynamical properties of topological dynamical systems. In this paper, we prove two theorems of a combinatorial nature about the recurrence of generalized Morse sequences, as defined by Keane (1968). These theorems are a strengthening of some of our earlier results. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Topological dynamical system; Recurrence; Generalized Morse sequence; IP set

AMS classification: 54H15; 54H20

1. Introduction

A topological dynamical system (X, T) consists of a topological space X and a homeomorphism $T : X \to X$. For a natural number $n \in \mathbb{N}$, T^n will denote the composition of T with itself n times and T^{-n} will denote $(T^{-1})^n$. Given a topological dynamical system (X, T), let $E(X)$ be the closure of the set $\{T^n\colon n \in \mathbb{Z}\}$ under the product topology of X^X, where $\mathbb{Z}$ denotes the set of integers. Let $H(X) = E(X) - \{\text{isolated points of } E(X)\}$. $E(X)$ and $H(X)$ are called the enveloping semigroup of X and the asymptotic enveloping semigroup of X, respectively, and both are semigroups under composition of functions [3,8].

An IP set in $\mathbb{N}$ (or $\mathbb{Z}$) is a subset P of $\mathbb{N}$ (or $\mathbb{Z}$) which coincides with the set of finite sums $p_{n_1} + \cdots + p_{n_k}$, $n_1 < \cdots < n_k$, taken from a sequence $(p_n)_{n=1}^{\infty}$ of distinct elements in $\mathbb{N}$ (or $\mathbb{Z}$). The sequence $(p_n)_{n=1}^{\infty}$ is called the generating sequence of P.

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0166-8641/99/$ – see front matter
PII: S0166-8641(98)00108-4

Given a topological dynamical system (X, T) and an IP subset P of $\mathbb{Z}$, an IP cluster point (IPCP) f along P is an element of $H(X)$ satisfying the property: For every neighborhood U of f (in the product topology), $\{n \in P\colon T^n \in U\}$ contains an IP set.

A function f is an IPCP along an IP set P if and only if f is a pointwise limit of idempotents in $E(X)$ [8]. This result is one of several in a vein that ties dynamical properties of topological dynamical systems with combinatorial properties (for further literature on the aforementioned connection, please consult any of [1,2,4–7]).

In [9], the authors compute the set of IPCPs of a class of dynamical systems known as generalized Morse flows [10]. Some combinatorial results follow from this computation. In this paper, we show that in a certain sense, these results can be strengthened.

2. Preliminaries

Let $\Omega = \{0, 1\}^{\mathbb{Z}}$ and $X = \{0, 1\}^{\mathbb{N}\cup\{0\}}$. Let $\mathcal{B}$ be the set of finite blocks of 0's and 1's. For $b \in \mathcal{B}$, Ω, or X, denote the ith slot of b by b_i, and define $\overline{b}$, the dual of b, as

$$\overline{b}_i = \begin{cases} 0 & \text{if } b_i = 1, \\ 1 & \text{if } b_i = 0. \end{cases}$$

For $z \in \Omega$ (or X), if $a \in \mathbb{Z}$ (or $\mathbb{N}$) and $n \in \mathbb{N}$ set $z(a, n) = z_a z_{a+1} \cdots z_{a+n-1}$. Let $l(b)$ be the length of block b, and when $l(b)$ is even, set

$$[b] = \left\{\xi \in \Omega\colon \xi\left(-\frac{l(b)}{2}, l(b)\right) = b\right\},$$

the cylinder set associated with b.

Define on $\mathcal{B}$ an operation "$\times$" on blocks by first setting $b \times 0 = b$ and $b \times 1 = \overline{b}$ for $b \in \mathcal{B}$ and then for a fixed block $c = c_0 c_1 \cdots c_n \in \mathcal{B}$, define $b \times c$ as $b \times c_0 + b \times c_1 + \cdots + b \times c_n$, where "+" denotes concatenation. If $c_0 = 0$, then $b \times c$ is simply an extension of b. The operation $\times$ just defined is associative, and if $b^0, b^1, b^2, \ldots$ are chosen so that $b^i_0 = 0$ and $l(b^i) > 1$ for every $i \geqslant 1$, then the sequence $b^0 \times b^1 \times b^2 \times \cdots$ is well defined and infinite. Such a sequence shall be called a recurrent sequence. It is shown in [10] that a recurrent sequence is periodic if and only if there exists a $k \in \mathbb{N} \cup \{0\}$ such that $b^k \times b^{k+1} \times \cdots$ equals either $00000\ldots$ or $010101\ldots$.

For $z \in \Omega$, let

$$\mathcal{B}_z = \{b \in \mathcal{B}\colon \exists k \in \mathbb{Z} \text{ such that } z(k, l(b)) = b\}.$$

Similarly, for $x \in X$, let

$$\mathcal{B}_x = \{b \in \mathcal{B}\colon \exists k \in \mathbb{N} \text{ such that } z(k, l(b)) = b\}.$$

For $x \in X$, let

$$O_x = \{\omega \in \Omega\colon \mathcal{B}_\omega \subseteq \mathcal{B}_x\}.$$

Let S be the shift map on Ω defined by

$$S(\xi)(n) = \xi(n + 1) \quad \text{for } \xi \in \Omega \text{ and } n \in \mathbb{Z}.$$

The orbit of a point $\xi \in \Omega$ is $O(\xi) = \{S^n(\xi)\colon n \in \mathbb{Z}\}$.

In [10], Lemma 4 shows that every recurrent sequence x can be extended to the left in two and only two ways so that the extension is in O_x. Furthermore, the two extensions are dual of each other at the negative indices. We will denote the elements of Ω which coincide with x at all non-negative indices by ω_x and ν_x. We will write $\omega_x = \omega$ and $\nu_x = \nu$ when dropping the index x does not lead to confusion. The following proposition follows easily:

Proposition 2.1. *Let $c^t = b^0 \times b^1 \times \cdots \times b^t$ and denote the length of c^t by l_t. Set $C = \{N\colon 1 \leqslant N \leqslant t$ and b^N ends with a $1\}$. If $|C|$ is odd, then $\omega(-l_t, 2l_t) = \overline{c^t} + c^t$ and if $|C|$ is even, then $\omega(-l_t, 2l_t) = c^t + c^t$.*

For the rest of the paper, $x = b^0 \times b^1 \times b^2 \cdots$ will denote a fixed nonperiodic recurrent sequence in X, and ω and ν will be as above. Set $M = \overline{O(\omega)}$, the orbit closure of ω, and S the shift on M. Finally, set $l_n = l(b^0 \times b^1 \times b^2 \times \cdots \times b^n)$.

3. Recurrent sequences and IP sets

The following theorem summarizes the main results in [9]. The technique used in the proof of the theorem requires factoring M onto a simpler space, computing the set of idempotents there, and then working back up to (M, S).

Theorem 3.1. *Let A, B, C, D be the elements of $H(M)$ defined as the identity off the orbits of $\omega, \overline{\omega}, \nu$ or $\overline{\nu}$, and otherwise defined as follows:*

$$
\begin{aligned}
A &: \omega \to \omega,\ \overline{\omega} \to \overline{\omega},\ \overline{\nu} \to \omega,\ \nu \to \overline{\omega},\\
B &: \omega \to \overline{\nu},\ \overline{\omega} \to \nu,\ \overline{\nu} \to \overline{\nu},\ \nu \to \nu,\\
C &: \omega \to \omega,\ \overline{\omega} \to \overline{\omega},\ \overline{\nu} \to \overline{\omega},\ \nu \to \omega,\\
D &: \omega \to \nu,\ \overline{\omega} \to \overline{\nu},\ \overline{\nu} \to \overline{\nu},\ \nu \to \nu.
\end{aligned}
$$

(A, B, C and D are now well defined as they commute with S on the orbits of $\omega, \overline{\omega}, \nu$ or $\overline{\nu}$.) Then A, B, C and D are the only IPCPs of M, and when (M, S) is viewed as an $\mathbb{N}$-action (which means that only positive powers of S are considered), then C and D are the only IPCPs of M.

The following corollaries follow readily from Theorem 3.1:

Corollary 3.2. *Let g be a map in $E(M)$ such that $g(\omega) \in \{\overline{\omega}, \overline{\nu}\}$. Then there exists a neighborhood U of g such that $\{n \in \mathbb{N}\colon S^n \in U\}$ does not contain an IP set.*

Corollary 3.3. *Let g be a map in $E(M)$ such that $g(\omega) \in \{\omega, \nu\}$ and such that g interchanges η and $\overline{\eta}$ for some $\eta \notin O(\omega) \cup O(\nu) \cup O(\overline{\omega}) \cup O(\overline{\nu})$. Then there exists a neighborhood U of g such that $\{n \in \mathbb{N}\colon S^n \in U\}$ does not contain an IP set.*

A natural question arises: Can we be more specific about which neighborhoods U satisfy the conclusions of Corollaries 3.2 and 3.3? The surprising answer to this question comes in the form of the following two theorems, the main results of this paper.

Theorem 3.4. *Let g be a map in $E(M)$ such that $g(\omega) \in \{\overline{\omega}, \overline{\nu}\}$. Let t be* any *positive integer and let $c^t = b^1 \times b^2 \times \cdots \times b^t$. Let U be* any *neighborhood of g which satisfies the property $\xi \in U \Rightarrow \xi(-l(c^t), 2l(c^t)) \in \{\overline{c^t} + \overline{c^t}, c^t + \overline{c^t}\}$. Then $\{n \in \mathbb{N}\colon S^n \in U\}$ does not contain an IP set.*

Theorem 3.5. *Let g be a map in $E(M)$ such that $g(\omega) \in \{\omega, \nu\}$ and such that g interchanges η and $\overline{\eta}$ for some $\eta \notin O(\omega) \cup O(\nu) \cup O(\overline{\omega}) \cup O(\overline{\nu})$. Let t be* any *positive integer and let $c^t = b^1 \times b^2 \times \cdots \times b^t$. Let U be* any *neighborhood of g which satisfies the property $\xi \in U \Rightarrow \xi(-l(c^t), 2l(c^t)) \in \{c^t + c^t, \overline{c^t} + c^t\}$. Let V be a neighborhood of g which satisfies the property $\xi \in V \Rightarrow \xi(\eta)$ and $g(\eta)$ agree on at least the slots 0 and 1. Then $\{n \in \mathbb{N}\colon S^n \in U$ and $S^n \in V\}$ does not contain an IP set.*

We now set out to prove the above two theorems. Recall that ω has forward orbit $b^0 \times b^1 \times b^2 \times \cdots$. Let i^k denote a location in b^k of a 0, and j^k a location of a 1. That is, for $b^k = 0b_1^k b_2^k \cdots b_{l(b^k)-1}^k$, $b_{i^k}^k$ is always a zero and $b_{j^k}^k$ is always a 1. We will further distinguish between positions of 0's and 1's as follows:
- i_0^k are locations of zeros which are preceded by a zero;
- i_1^k are locations of zeros which are preceded by a one;
- j_0^k are locations of ones which are preceded by a zero;
- j_1^k are locations of ones which are preceded by a one.

For example, if $b^k = 00110$ then $\{i^k\} = \{1, 4\}$ with $\{i_0^k\} = \{1\}$, $\{i_1^k\} = \{4\}$, and $\{j^k\} = \{2, 3\}$ with $\{j_0^k\} = \{2\}$, $\{j_1^k\} = \{3\}$.

We want to discuss return times of ω to neighborhoods of itself and to neighborhoods of its dual. So we will look at $\{n\colon S^n\omega \in [B]\}$ where B is the block $c^t + c^t$, $\overline{c^t} + c^t$, $\overline{c^t} + \overline{c^t}$, or $c^t + \overline{c^t}$ for some fixed t (see Proposition 2.1). We will say 'B appears at position n' if $S^n\omega \in [B]$. Rename c^t as b, and renumber so that the forward orbit of ω is $b \times b^2 \times b^3 \times \cdots$. Thus B can be any element of $\widetilde{B} = \{b + b, \overline{b} + b, \overline{b} + \overline{b}, b + \overline{b}\}$.

Lemma 3.6. *Let $A_k = b \times b^2 \times \cdots \times b^k$ and*

$$p_k = \big|\{n\colon 2 \leqslant n \leqslant k - 1 \text{ and } b^n \text{ ends with a } 1\}\big|.$$

In A_k with p_k even, $b + b$ appears at locations $i_0^k l_{k-1}$

$$\overline{b} + b \text{ appears at locations } i_1^k l_{k-1},$$
$$\overline{b} + \overline{b} \text{ appears at locations } j_1^k l_{k-1},$$
$$b + \overline{b} \text{ appears at locations } j_0^k l_{k-1}.$$

In A_k with p_k odd,

$$b + b \text{ appears at locations } i_1^k l_{k-1},$$

$\overline{b}+b$ appears at locations $i_0^k l_{k-1}$,

$\overline{b}+\overline{b}$ appears at locations $j_0^k l_{k-1}$,

$b+\overline{b}$ appears at locations $j_1^k l_{k-1}$.

Proof. Write A_k as $A_{k-1}+A_{k-1}\times b_1^k+A_{k-1}\times b_2^k+\cdots+A_{k-1}\times b^k_{l(b^k)-1}$. Because each b^n, $1\leqslant n\leqslant k$, begins with a zero, A_{k-1} always begins with block b. If p_k is even (odd), A_{k-1} will end with b $(\overline{b})$. □

The proofs of the following two lemmas are straightforward.

Lemma 3.7. *Take $B\in\widetilde{B}$. If B appears at location α in A_k, then it also appears at locations $i^{k+1}l_k+\alpha$ in A_{k+1}.*

Lemma 3.8. *Take $B\in\widetilde{B}$. If B appears at location α in A_k, then $\overline{B}$ appears at locations $j^{k+1}l_k+\alpha$ in A_{k+1}.*

We will use these three lemmas to describe the locations of B. Lemma 3.6 says where a block first appears and the next two lemmas describe how locations are carried forward by the repetition of the construction. For example, for p_k even, some locations of $\overline{b}+\overline{b}$ are $j_1^k l_{k-1}$, $j_1^k l_{k-1}+i^{k+1}l_k$, $j_1^k l_{k-1}+i^{k+1}l_k+j^{k+2}l_{k+1}+j^{k+3}l_{k+2}$. We leave the proof of the following proposition to the reader:

Proposition 3.9. *Let $m\in N$ be arbitrary and $k_1<k_2<\cdots<k_m$. Then*

$$\left\{\sum_{r=1}^{m} a_{k_r}l_{k_r}:\ \left|\left\{a_{k_r}:\ a_{k_r}\in\{j^{k_r}\}\right\}\right| \text{ is odd, and}\right.$$
$$\left. a_{k_1}\in\{j_1^{k_1}\}\text{ for } p_{k_1}\text{ even or } a_{k_1}\in\{j_0^{k_1}\}\text{ for } p_{k_1}\text{ odd}\right\}$$

are locations of $\overline{b}+\overline{b}$.

$$\left\{\sum_{r=1}^{m} a_{k_r}l_{k_r}:\ \left|\left\{a_{k_r}:\ a_{k_r}\in\{j^{k_r}\}\right\}\right| \text{ is odd, and}\right.$$
$$\left. a_{k_1}\in\{j_0^{k_1}\}\text{ for } p_{k_1}\text{ even or } a_{k_1}\in\{j_1^{k_1}\}\text{ for } p_{k_1}\text{ odd}\right\}$$

are locations of $b+\overline{b}$.

$$\left\{\sum_{r=1}^{m} a_{k_r}l_{k_r}:\ \left|\left\{a_{k_r}:\ a_{k_r}\in\{j^{k_r}\}\right\}\right| \text{ is even, and}\right.$$
$$\left. a_{k_1}\in\{i_0^{k_1}\}\text{ for } p_{k_1}\text{ even or } a_{k_1}\in\{i_1^{k_1}\}\text{ for } p_{k_1}\text{ odd}\right\}$$

are locations of $b+b$.

$$\left\{\sum_{r=1}^{m} a_{k_r} l_{k_r}:\ \left|\{a_{k_r}:\ a_{k_r} \in \{j^{k_r}\}\}\right| \text{ is even, and}\right.$$

$$\left. a_{k_1} \in \{i_1^{k_1}\} \text{ for } p_{k_1} \text{ even or } a_{k_1} \in \{i_0^{k_1}\} \text{ for } p_{k_1} \text{ odd}\right\}$$

are locations of $\overline{b} + b$.

Because of possible symmetry within a block, the above description may not include all locations of the block B. For instance, if $b = 000$, $b^2 = 01000$ then $b \times b^2 =$ 000 111 000 000 000 and the block $b + b = 000\ 000$ appears not only at $3l_1$ and $4l_1$ but also at $3l_1 + 1$ and $3l_1 + 2$. However, if we make $b = 0$ then this description is complete and any first block b can be rewritten as $0 \times b$.

We can now use this information about the position of blocks to prove Theorems 3.4 and 3.5. In both cases we assume an IP set exists and use the lemmas to find a subsequence of the IP set which contradicts the assumption.

Proof of Theorem 3.4. It is enough to show the result for $U = \{\xi \in E(M): \xi(\omega) = [B]\}$, where $[B] = [11]$ or $[01]$. Any other neighborhood will be contained in one of these.

Assume $\{n \in \mathbb{N}: S^n\omega \in [B]\}$ does contain an IP set P with generators $\{p_i\}$. Recall that each first block b can be rewritten as $0 \times b$; so by Proposition 3.9, we know that every $p \in P$ can be written as a summation which has an odd number of j's for coefficients. This means that any two generators p_s and p_r must have at least one common term (i.e., there exists k such that $a_k l_k$ and $b_k l_k$ are terms in p_s and p_r, respectively, with a_k and b_k nonzero), because otherwise $p_r + p_s$ will have an even number of j-coefficients. Since each p_s has only finitely many terms, we can find an infinite subset of $\{p_i\}$ with a common term. In fact we can find an infinite subset of $\{p_i\}$ (again denoted by $\{p_i\}$) with a common beginning $e_1 = a_0 + a_1 l_1 + \cdots + a_k l_k$. Write $p_1 = e_1 + r_1$. Let $d =$ degree of $p_1 = \max\{k:$ coefficient of l_k in p_1 is nonzero$\}$. It is possible to find e_2, with degree $>$ d, which is a common beginning in an infinite subset of the remaining $\{p_i\}$, and to pick $p_2 = e_2 + r_2$. Continue in this manner to find a subsequence $\{p_{i_r}\}$ whose terms have a longer and longer common beginning. Since $\sum_{r=2}^{d+1} p_{i_r}$ is a location of 11 (or 01) it can be written in summation form with an odd number of j-coefficients. Yet because of their common beginnings, $\min\{k:$ coefficient of l_k in $\sum_{r=2}^{d+1} p_{i_r}$ is nonzero$\} > d$. Thus p_1 and $\sum_{r=2}^{d+1} p_{i_r}$ have no terms in common and $p_1 + \sum_{r=2}^{d+1} p_{i_r}$ has an even number of j-coefficients which contradicts it being a position of 11 (or 01). □

Proof of Theorem 3.5. Assume without loss of generality that $\eta_0 = 0$, otherwise change η to $\overline{\eta}$. It is enough to prove the theorem for $U = \{\xi \in E(M): \xi(\omega) \in [B_\omega]\}$, where $[B_\omega] = [00]$ or $[10]$, and for $V = \{\xi \in E(M): \xi(\eta) \in [B_{\overline{\eta}}]\}$ where $[B_{\overline{\eta}}] = [11]$ or $[01]$.

Assume $\{n \in \mathbb{N}: S^n\omega \in [B_\omega]$ and $S^n\eta \in [B_{\overline{\eta}}]\}$ contains an IP set. In a manner similar to that used in the last proof, pick a subsequence $\{q_i\}$ from this IP set, each the sum over disjoint collections of generators, so that $q_i = \sum_{k=t_i}^{m_i} a_k l_k$ where

$$t_i = \min\{k\colon \text{ coefficient of } l_k \text{ in } q_i \text{ is nonzero}\},$$
$$m_i = \max\{k\colon \text{ coefficient of } l_k \text{ in } q_i \text{ is nonzero}\}, \quad \text{and} \quad m_i < t_{i+1}.$$

Note that the q_i have an even number of j-coefficients.

Next take $\{K_d\}$, a sequence in $\mathbb{N}$ going to infinity, so that $\operatorname{Lim} S^{K_d}\omega = \eta$ and so that $\omega(K_d - L, 2L + 1) = \eta(-L, 2L + 1)$ where $L > \sum_{i=1}^{d} q_i$. This is possible because $\eta \in \overline{O(\omega)}$. Because of the form of η, K_d can be written with an even number of j-coefficients. Because ω and η agree at slots $K_d - L$ through $K_d + L$, and because of the the form of $\overline{\eta}$ and Proposition 3.9, $K_d + q_i$ can be written with an odd number of j-coefficients.

The rest of the proof explores the form K_d must have. We first find a subsequence of $\{K_d\}$ which has longer and longer common beginning terms. We will say that a summand $b_j l_j$ in K_d is affected by $a_i l_i$ if either $i = j$ or $i < j$ and K_d also includes nonzero coefficients for all the terms $l_i, l_{i+1}, \dots, l_{j-1}$, so that when summed with $a_i l_i$ they yield a term $c_j l_j$. Let K_d^i be those summands in K_d which are affected by the terms in q_i. Then there must be some overlap between the highest degrees term in K_d^1 and the lowest degrees term in K_d^d, otherwise since $K_d + q_1$ and $K_d + q_d$ have an odd number of j-coefficients, $K_d + q_1 + q_d$ will have an even number of j-coefficients. Thus K_d includes terms $\sum_{k=m_1}^{t_d} a_k l_k$ and in fact this is true for any K_n, $n \geqslant d$. We can thus find a subsequence, denoted by $\{K_r\}$, such that for all r, all coefficients of $l_0, l_1, \dots, l_{t_d}$ are identical; and a further subsequence with longer and longer common beginnings can be constructed. Then we can find a fixed $\gamma \in \mathbb{N}$ such that $K_r + \gamma$ only has terms of degree higher than l_{t_d}, so we can write $K_r + \gamma = l_{t_d} \cdot \alpha_r$ where $\alpha_r \in \mathbb{N}$.

We will next see what this structure forces on η. Let $C = b \times b^2 \times \cdots \times b^{t_d}$. Then ω can be written as C's and $\overline{C}$'s concatenated together as b^i $(i > t_d)$ dictates. So $S^{K_r+\gamma}\omega(-l_{t_d}, 2l_{t_d} + 1)$ is either $C + C$, $C + \overline{C}$, $\overline{C} + C$, or $\overline{C} + \overline{C}$ which agrees with $\omega, v, \overline{\omega}$, or $\overline{v}$ by Proposition 2.1. Note that d is arbitrary and $l_{t_d} \to \infty$ as $r \to \infty$.

Since $S^{K_r}\omega \to \eta$, $S^{K_r+\gamma}\omega \to S^\gamma \eta$ and we have shown that η is either $\omega, v, \overline{\omega}$, or $\overline{v}$ shifted by γ. This contradicts our original assumption on η. □

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