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Rank one and loosely Bernoulli actions in $\mathbb{Z}^d$

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Abstract. We define rank one for $\mathbb{Z}^d$ actions and show that those rank one actions with a certain tower shape are loosely Bernoulli for $d \geq 1$. We also construct a zero entropy $\mathbb{Z}^2$ loosely Bernoulli action with a zero entropy, ergodic, non-loosely Bernoulli one-dimensional subaction.

1. *Introduction*

The idea of loosely Bernoulli for one-dimensional actions was first introduced by Feldman **[F]** and Katok **[K]** in the late 1970's in their study of K-automorphisms which were not Bernoulli. Loosely Bernoulli is the analog of very weak Bernoulli but with the $\overline{d}$-metric replaced by a weaker metric, now called the $\overline{f}$-metric. In **[ORW]**, Ornstein, Rudolph and Weiss use this $\overline{f}$-metric to define finitely fixed processes and show that finitely fixed is equivalent to loosely Bernoulli. They also show that all finite rank transformations are loosely Bernoulli. The $\overline{f}$-metric was defined for higher-dimensional actions by Hasfura-Buenago **[H]**. We use this generalized definition to show that $\mathbb{Z}^d$ rank one actions whose towers are of a certain shape are loosely Bernoulli. Our approach is a traditional 'nesting argument'. The new ingredient lies in the complications introduced by working in many dimensions. The higher-dimensional definition of the $\overline{f}$-metric requires that the relative configuration of points in the d-dimensional lattice not be changed too much. The nesting argument we provide here is designed to address this geometric issue which does not arise in the one-dimensional case. We then use this result and a one-dimensional construction of Feldman **[F]** to construct a $\mathbb{Z}^2$ zero entropy, loosely Bernoulli action with a one-dimensional subaction which is ergodic, has zero entropy and is not loosely Bernoulli.

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2. *Rank one and loosely Bernoulli in $\mathbb{Z}^d$*

2.1. *Background.* Let $(X, \mathcal{A}, \mu)$ be a Lebesgue probability space. Take T to be an ergodic $\mathbb{Z}^d$ action on $(X, \mathcal{A}, \mu)$. We can think of T as being generated by d commuting measure-preserving one-dimensional transformations on X, $\{T_{\vec{e}_1}, \ldots, T_{\vec{e}_d}\}$, where $\{\vec{e}_1, \ldots, \vec{e}_d\}$ is the standard basis for $\mathbb{Z}^d$. Then $T_{\vec{v}}(x) = T_{\vec{e}_1}^{v_1} \circ \cdots \circ T_{\vec{e}_d}^{v_d}(x)$, where $\vec{v} = (v_1, \ldots, v_d)$. We call $((X, \mathcal{A}, \mu), T)$ a $\mathbb{Z}^d$-dynamical system. Often we will simply write (X, T).

Let P be a finite label set, or equivalently, a finite measurable partition $P = \{p_1, \ldots, p_h\}$ on X. (T, P) is then the usual process associated with T and the partition P. Set $\|\vec{v}\| = \max\{|v_i| : 1 \le i \le d\}$, and for $n \in \mathbb{N}$, $B_n = \{\vec{v} \in \mathbb{Z}^d : \|\vec{v}\| \le n\}$. For each x we can then define its P_n-name to be $P_n(x) : B_n \to P$ by $P_n(x)(\vec{v}) = i$ if $T_{\vec{v}}(x) \in p_i$.

In order to define a loosely Bernoulli process we start with $\pi : B_n \to B_n$, a permutation of the indices in B_n, and define a size for this permutation. This idea is defined and extended in [H] and [KR].

Definition 2.1.1. Let $\pi : B_n \to B_n$ be a permutation of the indices of B_n. We say π is of size ϵ, denoted by $m(\pi) < \epsilon$, if there exists a subset S of B_n satisfying:
(i) $|S| > (1 - \epsilon)|B_n|$, where $|S|$ is the cardinality of the set S;
(ii) $\|\pi\vec{u} - \pi\vec{v} - (\vec{u} - \vec{v})\| < \epsilon\|\vec{u} - \vec{v}\|$ for every $\vec{u}, \vec{v} \in S$.

Definition 2.1.2. Given two P_n-names η and ξ, we define the $\overline{f}_n$-distance between them to be $\overline{f}_n(\eta, \xi) = \inf\{\epsilon > 0$: there exists a permutation π of B_n such that:
(i) $m(\pi) < \epsilon$;
(ii) $\overline{d}(\eta \circ \pi, \xi) < \epsilon\}$.
Here $\overline{d}(.,.)$ denotes the Hamming metric which simply gives the proportion of locations of B_n on which the two names disagree.

Informally, we will think of π as rearranging the name η to make it $\overline{d}$ close to the name ξ and we will often refer to π as acting on a name instead of the (technically correct) set of indices. If π, η, and ξ satisfy (ii) of the above definition we say π matches a $(1 - \epsilon)$-proportion of η and ξ.

Intuitively, a zero entropy loosely Bernoulli process has one name up to $\overline{f}$. Formally,

Definition 2.1.3. A zero entropy process (T, P) is loosely Bernoulli (LB) iff for any $\epsilon > 0$ there exists an integer N_ϵ such that for any $n \ge N_\epsilon$ and ϵ-a.e. atoms ω and ω' of $\bigvee_{\vec{v} \in B_n} T_{\vec{v}} P$,

$$\overline{f}_n(\omega, \omega') < \epsilon.$$

Definition 2.1.4. We say $((X, \mathcal{A}, \mu), T)$ is LB if for every partition P of X, (T, P) is LB.

2.2. *Square rank one transformations are LB*

Definition 2.2.1. $((X, \mathcal{A}, \mu), T)$ is a $\mathbb{Z}^d$ rank one transformation if there exists a sequence of sets $F_i \subset X$ and a Følner sequence D_i of subsets of $\mathbb{Z}^d$ such that for each i, $\{T_{\vec{v}} F_i\}$ are pairwise disjoint for $\vec{v} \in D_i$ and the partition $P_i = \{T_{\vec{v}} F_i : \vec{v} \in D_i, X - \cup_{\vec{v} \in D_i} T_{\vec{v}} F_i\}$ converges to $\mathcal{A}$ as $i \to \infty$.

In this paper we restrict our attention to rank one transformations with a special tower shape.

Definition 2.2.2. $((X, \mathcal{A}, \mu), T)$ is a $\mathbb{Z}^d$ square rank one transformation if it is rank one and there exists an $\alpha \geq 1$ such that for all i, the set D_i of Definition 2.2.1 satisfies:
(1) D_i is a rectangle of dimensions $\{l_1^i, \ldots, l_d^i\}$;
(2) if $s^i = \min\{l_k^i\}$ and $b^i = \max\{l_k^i\}$, then $s^i/b^i \geq 1/\alpha$.

For each i, $\cup_{\vec{v} \in D_i} T_{\vec{v}} F_i$ will be called the ith-tower and will be denoted by τ_i. A $\mathbb{Z}^d$ transformation T is then square rank one if there is a sequence of towers τ_i that converge to $\mathcal{A}$, and these towers are rectangles which are close in geometry to a square of side b^i. Any transformation which is rank one is ergodic and has zero entropy **[Fr, PR]**.

In the following we will assume that $((X, \mathcal{A}, \mu), T)$ is square rank one and fix $\epsilon > 0$. We need to show that for an arbitrary partition Q, (T, Q) satisfies Definition 2.1.3. We will first establish two technical lemmas and then proceed to prove the result in the subsequent theorem. Both lemmas are designed to show that given two names satisfying certain properties, we can always match a fixed proportion. The theorem shows that we can apply the lemmas many times to construct a permutation which matches all but ϵ of the two names. Neither lemma requires (2) of Definition 2.2.2. This condition is used to show that the permutation defined in the theorem is small. First, a definition which will simplify our notation in the following results.

Definition 2.2.3. Given a rectangle $R \subset \mathbb{Z}^d$ of size $l_1 \times \cdots \times l_d$, the ϵ-interior of R is the collection of indices in R which are at least a distance ϵl_k from the kth edge of R. The ϵ-collar of R is then the complement of the ϵ-interior and corresponds to the set of indices within ϵl_k of the kth edge in the boundary of R.

Denote the volume of D_i by $v_i = l_1^i \times \cdots \times l_d^i$ and notice that the volume of the ϵ-interior of D_i is $(1-2\epsilon)l_1^i \times \cdots \times (1-2\epsilon)l_d^i = (1-2\epsilon)^d v_i$. Let $\overline{D_i}$ be the rectangle of size $2l_1^i \times \cdots \times 2l_d^i$, centered at the orgin unless otherwise specified.

In the following lemma we show that given two words of size n which are full of k-towers, τ_k, we can be sure of finding a permutation which will match a certain percentage of these. The proof rests on counting occurrences of τ_k in the two words ω and ω'. The rough idea is to first count the τ_k's in ω which are not too close to the edge. Of these τ_k we count those for which there is a τ_k in a close-by location in ω'. We then match a subset of this last collection of towers.

LEMMA 2.2.4. (THE MATCHING LEMMA) *Let ϵ be fixed and set $a = (1-2\epsilon)^d(1-\epsilon)\epsilon^d$. Given k such that $\mu(\tau_k) > 1 - \epsilon/8$, let n be so large that:*
(1) $|\{\vec{v} \in B_n : T_{\vec{v}}(x) \in \tau_k\}|/|B_n| > 1 - \epsilon/4$ *for ϵ-a.e. $x \in X$;*
(2) $n^d/v_k > 1/\epsilon^d(1-\epsilon)$*;*
(3) $(n - b^k)^d/n^d > 1 - \epsilon/4$.
Let ω, ω' be $(P_k)_n$-names of two points x and x' given by (1). There exists a permutation $\pi : B_n \to B_n$ such that $d(\omega \circ \pi, \omega') < 1 - a$ where the action of π can be described as follows:
(A) *π translates the indices of B_n by a vector $\vec{v}$ with $\|\vec{v}\| < b^k$, except for those indices i for which $i + \vec{v} \notin B_n$. On these, $\pi(i)$ is defined to be one of the indices vacated by the translation.*

(B) *For a subset of the τ_k's occurring in ω, π moves the ϵ-interiors of these towers by an additional amount which can vary for each tower but is always less in magnitude than ϵb^k. The resulting location of the ϵ-interiors of these towers matches perfectly with the corresponding interior of a τ_k in ω'.*

Proof. Consider the n-name of a point x satisfying (1). At least $n^d(1-\epsilon/4)$ of the indices in B_n correspond to locations of τ_k-towers. By (3), at least $n^d(1-\epsilon/2)$ of the indices in B_n correspond to locations of τ_k-towers which intersect $B_{(n-4b^k)} + (2b^k, \dots, 2b^k)$ and the number of these towers must be at least $n^d(1-\epsilon/2)/v_k$.

Now fix ω, ω' to be $(P_k)_n$-names of two points x and x' satisfying (1). We want to construct a permutation π which we can think of as changing ω to be $\bar{d}$-close to ω'. Consider the occurrences of τ_k in ω which intersect the $B_{(n-4b^k)} + (2b^k, \dots, 2b^k)$. Enumerate these occurrences by $\{j\}$ and let $\{j(1)\}$ be the first lexicographic index in B_n at which the jth τ_k-tower occurs. Now consider the box $\overline{D_k}$ centered at $j(1)$ in ω'. Notice this box is entirely contained in B_n. We now count the number of $\overline{D_k} + j(1)$ which do not contain the first lexicographic occurrence of a τ_k-tower in ω'. Note that this occurs exactly when $(D_k + j(1))$ in ω' does not intersect any portion of a τ_k-tower. Since there are at most $(\epsilon/4)n^d$ locations in B_n corresponding to indices of ω' not in a τ_k tower and a D_k rectangle has size v_k, the number of these $\overline{D_k} + j(1)$ which do not contain a first lexicographic occurrence of a τ_k tower is at most $(\epsilon/4)n^d/v_k$.

Putting the estimates of the last two paragraphs together, the number of τ_k towers in ω which intersect $B_{(n-4b^k)} + (2b^k, \dots, 2b^k)$ and for which ω' has the first lexicographic occurrence of a τ_k tower in the corresponding $\overline{D_k}$ box is at least $n^d(1-\epsilon)/v_k$. Let $\vec{u}_j$ be the vector which begins at $j(1)$ and ends at the first location of this corresponding τ_k tower in ω'. If there is more than one corresponding tower, choose the tower which lexicographically appears first. Note that $\vec{u}_j \in \overline{D_k}$.

Next, divide $\overline{D_k}$ into $(1/\epsilon)^d$ boxes of size $2\epsilon l_1^k \times \cdots \times 2\epsilon l_d^k$. Since $(n^d/v_k)(1-\epsilon) > (1/\epsilon)^d$ by assumption (2), there must be at least one sub-box of $\overline{D_k}$ with at least two $\vec{u}_j$ in it. Let $\vec{v}$ be the midpoint of the sub-box with the most $\vec{u}_j$ in it and notice this sub-box must contain at least $(n^d/v_k)(1-\epsilon)\epsilon^d$ of the $\vec{u}_j$'s.

We will now define π, a permutation of B_n for which $\bar{d}(\omega \circ \pi, \omega') < 1-a$. If $i \in B_n$ is such that $i + \vec{v} \in B_n$, then $\pi(i) = i + \vec{v}$. For those indices with $i + \vec{v} \notin B_n$, $\pi(i)$ is an arbitrarily choosen vector not in the range of the translation by $\vec{v}$. π will be further modified on the collection of indices corresponding to τ_k's for which $\|\vec{u}_j - \vec{v}\| < \epsilon b^k$. The ϵ-interiors of these towers are moved rigidly by an additional vector, $\vec{u}_j - \vec{v}$. Those points in the ϵ-collar which are in the range of this additional shift come in rectangular pieces. They are moved, in blocks, to the rectangular pieces left vacant by this last translation. Note that the ϵ-interiors of these towers have now been matched with their counterparts in ω'.

If we have matched the ϵ-interiors of h towers, then we have matched $h(1-2\epsilon)^d v_k$ indices which means we have matched a proportion $h(1-2\epsilon)^d v_k/n^d$ of B_n. We have by our previous calculation that $h > (n^d/v_k)(1-\epsilon)\epsilon^d$, so we have matched at least $(1-2\epsilon)^d(1-\epsilon)\epsilon^d = a$ of B_n. □

The nesting lemma which follows will assure us of being able to continue the process started in the matching lemma: if we have two n-words which disagree on a set of indices $G^c \subset B_n$ and we know that on this set G^c the two words are full of τ_k-towers, then we can construct a permutation of G^c which will match a certain percentage of these towers. We think of G as being the 'good' set on which matching has already been done and G^c as being the set on which we still need to match. The proof of the nesting lemma is similar to that of the matching lemma except now the 'edge' of G^c is more complicated than simply the edge of B_n. Thus when we count the number of towers which are not too close to the edge of G^c we must consider both the outside edge of B_n and the inside edges consisting of the boundary of G.

LEMMA 2.2.5. (THE NESTING LEMMA) *Let ϵ be fixed and set $a = (1-2\epsilon)^d(1-\epsilon)\epsilon^d$. Suppose that $n > k_{m-1} > k_{m-2} > \cdots > k_i$ and $\omega, \omega' \in \vee_{\vec{v}\in B_n} T_{\vec{v}}(P_{k_{i-1}})$ are given such that there exists $G \subset B_n$ which is a disjoint union of boxes with sides of length $(1-2\epsilon)l_1^{k_j}, \ldots, (1-2\epsilon)l_d^{k_j}$ for j between i and $m-1$, satisfying:*

(a) $\omega|_G = \omega'|_G$;

(b) $\epsilon < |G^c|/n^d < (1-a)^{m-i}$;

(c) $b^{k_j}/s^{k_{j+1}} < \epsilon^2/32d(m-1)$ *for* $i \le j \le m-2$;

(d) $b^{k_{m-1}}/n < \epsilon^2/32d(m-1)$.

Then if $k_{i-1} < k_i$ satisfies:

(e) $b^{k_{i-1}}/s^{k_i} < \epsilon^2/32d(m-1)$;

(f) $|i \in G^c : \tilde{\omega}(i) \in \tau_{k_{i-1}}|/|G^c| > 1-\epsilon/8$ *for* $\tilde{\omega} = \omega, \omega'$,

there is a permutation $\pi : B_n \to B_n$ such that:

(A) $\bar{d}(\omega \circ \pi, \omega') < (1-a)^{m-i+1}$;

(B) $\pi|_G = \mathrm{id}$;

(C) *π moves G^c by translation by a vector $\vec{v}$ with $\|\vec{v}\| \le b^{k_{i-1}}$, except for those $i \in G^c$ for which $i + \vec{v} \notin G^c$; there $\pi(i)$ is defined to be one of the postions left vacant by the translation. For a subset of the $\tau_{k_{i-1}}$'s occurring in ω, π is further defined to move their ϵ-interiors by an amount less than $\epsilon b^{k_{i-1}}$ so that the resulting location is exactly that of the interior of a $\tau_{k_{i-1}}$ in ω'.*

Proof. We first want to count the number of indices in G^c which lie in a $\tau_{k_{i-1}}$ and are not too close to the boundary of G^c. Note that the boundary of G^c consists of the edge of B_n and the edges of the boxes making up G. We know from (f) that at least $(1-\epsilon/8)|G^c|$ of the indices of G^c lie in k_{i-1}-towers. There are at most $(2d)(2b^{k_{i-1}})n^{d-1}$ indices that lie within $2b^{k_{i-1}}$ of the faces of B_n, which is at most $(2d)(2b^{k_{i-1}})n^{d-1}/|G^c|$ of G^c. Using (b) we see that this is less than $(2d)(2b^{k_{i-1}})/n\epsilon$, which by (e) is less than $\epsilon/8$ of G^c. The proportion of indices in G^c which lie within $2b^{k_{i-1}}$ of G is at most

$$\sum_{j=i}^{m-1} \frac{2(2b^{k_{i-1}}) \sum l_{p_1}^j \times \cdots \times l_{p_{d-1}}^j z_{k_j}}{|G^c|},$$

where z_{k_j} is the number of boxes in G of volume $(1-2\epsilon)l_j^1 \times \cdots \times (1-2\epsilon)l_j^d$ and the inside summation is over all possible combinations of $(d-1)$ elements choosen from $\{1, \ldots, d\}$. Note that for any j, z_{k_j} must be less than n^d/v_j. Thus the proportion of indices within $2b^{k_{i-1}}$ of G is

$$< \sum_{j=i}^{m-1} \frac{4b^{k_{i-1}} \sum l_{p_1}^j \times \cdots \times l_{p_{d-1}}^j n^d}{|G^c| v_j}.$$

Using that $v_j = l_1^j \times \cdots \times l_d^j$ we have this is

$$< \sum_{j=i}^{m-1} \frac{4b^{k_{i-1}} d n^d}{|G^c| s^{k_j}} \quad \text{which by (b) is}$$

$$< \sum_{j=i}^{m-1} \frac{4b^{k_{i-1}} d}{\epsilon s^{k_j}} \quad \text{which by (e) is}$$

$$< \sum_{j=i}^{m-1} \frac{4d\epsilon^2}{\epsilon 32 d(m-1)} < \frac{\epsilon}{8}.$$

Altogether, the number of indices in G^c which lie in a k_{i-1}-tower and are at least $2b^{k_{i-1}}$ from the boundary of G^c is at least $(1 - 3\epsilon/8)|G^c|$. Now follow the steps of the matching lemma: enumerate the above k_{i-1}-towers by $\{j\}$ and their first occurrences by $\{j(1)\}$. Consider the box $\overline{D}_{k_{i-1}}$ centered at $j(1)$ in ω'. At most $(\epsilon/8)|G^c|/v_{i-1}$ of these do not contain a first occurrence of a $\tau_{k_{i-1}}$. Thus we have at least $|G^c|(1-\epsilon)/v_{i-1}$ k_{i-1}-towers in ω with a corresponding $\tau_{k_{i-1}}$ in ω'. Label the $\vec{u}_j$'s as before, divide $\overline{D}_{k_{i-1}}$ as before and pick $\vec{v}$ as before. Define π similarly to the matching lemma, moving all of $|G^c|$ by a fixed vector and the ϵ-interiors of a subset of the $\tau_{k_{i-1}}$ by an additional amount less in magnitude than $\epsilon b^{k_{i-1}}$. We will be assured of matching the interiors of at least $(|G^c|(1-\epsilon)/v_{i-1})\epsilon^d$ k_{i-1}-towers, thus at least $(|G^c|(1-\epsilon)/v_{i-1})\epsilon^d(1-2\epsilon)^d v_{i-1}$ indices which is $(1-\epsilon)(1-2\epsilon)^d\epsilon^d = a$ of $|G^c|$. This yields (A) of the result, and (B) and (C) follow from the construction of π. □

Now we will prove the main result of this section. This involves three steps. First we carefully pick a sequence of tower sizes that assure that the n-names of most atoms will contain many copies of these towers. Next we apply one of the last two lemmas for each size tower and compose together the resulting permutations to yield a permutation on the n-name. Finally, we show this permutation satisfies Definition 2.1.2. It is in this last step that we will use the geometry of our rectangles.

THEOREM 2.2.6. *Square rank one implies LB.*

Proof. Assume $(X, \mathcal{A}, \mu), T$ is a square rank one $\mathbb{Z}^d$ transformation. Let Q be an arbitrary partition on X and consider (T, Q). Let ϵ be fixed and set $\overline{\epsilon} = \epsilon^2/16d^2\alpha^2$. Put $a = (1-2\overline{\epsilon})^d(1-\overline{\epsilon})\overline{\epsilon}^d$ and take m such that $(1-a)^{m-1} < \epsilon/2$.

We first choose an increasing sequence of towers $\{\tau_{k_i}\}$ by taking integers $0 < k_1 < \cdots < k_{m-1}$ so that:

(i) $b^{k_{i-1}}/s^{k_i} < \overline{\epsilon}^2/40d(m-1)$;

(ii) $\mu(\tau_{k_i}) > 1 - \overline{\epsilon}^2/80$.

Now pick $K > 0$ so that for every $n \geq K$:

(iii) $n^d/v_{k_{m-1}} > 1/\overline{\epsilon}^d(1-\overline{\epsilon})$;

(iv) $(n - 4b^{k_{m-1}})^d / n^d > 1 - \bar{\epsilon}/4$;

(v) for $(\epsilon/2)$-a.e. $x \in X$, for $i = 1, \dots, m-1$, $|\{\vec{v} \in B_n : T_{\vec{v}}(x) \in \tau_{k_i}\}|/n^d > 1 - \bar{\epsilon}^2/40$.

Choose $t \in \mathbb{N}$ such that $t > k_{m-1}$ and there exists $Q_t \subset \bigvee_{i=1}^t P_i$ such that $d(Q, Q_t) < \epsilon/8$. Use this with (v) above to find a set A with $\mu(A) \geq 1 - \epsilon$ and $n \in \mathbb{N}$ such that for $i = 1, \dots, m-1$,

$$\frac{|\vec{v} \in B_n : T_{\vec{v}}x \in \tau_{k_i}|}{|B_n|} > 1 - \frac{\bar{\epsilon}^2}{40},$$

and for all $x \in A$, the $(Q_t)_n$-name of x and the Q_n-name of x differ less than $\epsilon/4$ of the time.

Consider ω, ω', two $(\bigvee_{i=1}^t P_i)_n$-names of points in A. We will define a permutation $\pi : B_n \to B_n$ such that

$$\bar{d}(\omega \circ \pi, \omega') < \frac{\epsilon}{2} \quad \text{and} \quad m(\pi) < \epsilon. \tag{1}$$

We can then use this same π on the associated Q_t atoms and again obtain equation (1). Finally, we use this same π on the associated Q atoms to show that (T, Q) is LB.

First apply the matching lemma to ω and ω' with tower $\tau_{k_{m-1}}$ and $\epsilon = \bar{\epsilon}$ to obtain $\pi_{m-1} : B_n \to B_n$ such that (A) and (B) of that lemma hold.

Now set G_{m-1} to be the indices of the ϵ-interiors of the $\tau_{k_{m-1}}$ matched by π_{m-1}. If $|G_{m-1}|/n^d > 1 - \epsilon/2$ then we need only show $m(\pi) < \epsilon$ to complete this proof. Otherwise, in the nesting lemma set $G = G_{m-1}$, $k_i = k_{m-1}$, and $\epsilon = \bar{\epsilon}$. It is immediate that (b)–(e) hold, and (a) holds for $\omega \circ \pi_{m-1}, \omega'$. We now show that (f) is satisfied, i.e. that G^c is full of $\tau_{k_{m-2}}$-towers. We do this by first noting that ω and ω' are all but $\bar{\epsilon}^2/40$ covered by $\tau_{k_{m-2}}$-towers. When we applied the matching lemma we potentially increased this error with the rigid translations that we applied. We count the number of suspect indices, and remove them from consideration.

The first rigid translation by $\vec{v}_{m-1}$ may have affected the $\tau_{k_{m-2}}$ which intersect both the interior and exterior of $B_{(n-4b^{k_{m-1}})} + (2b^{k_{m-1}}, \dots, 2b^{k_{m-1}})$. The total number of indices in question is less than $(2d)n^{d-1}b^{k_{m-2}}$, so the proportion of B_n they take up is less than

$$2d\frac{b^{k_{m-2}}}{n} < 2d\frac{\bar{\epsilon}^2}{40d(m-1)} < 2\frac{\bar{\epsilon}^2}{40}.$$

After we throw these indices away, B_n is all but $3\bar{\epsilon}^2/40$ covered by $\tau_{k_{m-2}}$.

Secondly, we may have disturbed some k_{m-2}-towers when the $\bar{\epsilon}$-interiors of some of the k_{m-1}-towers were moved. The number of indices in question now is less than $2\sum_{i=1}^d l_i^{k_{m-2}}[\prod_{j\neq i} l_j^{k_{m-1}}] \times z_{m-1}$, where z_{m-1} is the number of k_{m-1}-towers matched at the last stage. Since $z_{m-1} < n^d/v_{m-1}$, we remove from consideration at most the proportion

$$\begin{aligned}\frac{2\sum_{i=1}^d l_i^{k_{m-2}} \prod_{j\neq i} l_j^{k_{m-1}} z_{m-1}}{n^d} &< \frac{2\sum_{i=1}^d l_i^{k_{m-2}} \prod_{j\neq i} l_j^{k_{m-1}}}{\prod l_j^{k_{m-1}}} < 2d\frac{b^{k_{m-2}}}{s^{k_{m-1}}} \\ &< 2d\frac{\bar{\epsilon}^2}{40d(m-1)} < 2\frac{\bar{\epsilon}^2}{40}.\end{aligned}$$

As a result, we have that B_n is all but $5\bar{\epsilon}^2/40$ covered by $\tau_{k_{m-2}}$ which are undisturbed by π_{m-1}. Now note that $n^d/|G_{m-1}^c| < 1/\bar{\epsilon}$ implies that the proportion of G_{m-1}^c not in a

k_{m-2}-tower is

$$\frac{5\bar{\epsilon}^2}{40}\frac{n^d}{|G^c_{m-1}|} < \frac{5\bar{\epsilon}^2}{40}\frac{1}{\bar{\epsilon}} = \frac{\bar{\epsilon}}{8}.$$

So assumption (f) of the nesting lemma holds. Apply the lemma to obtain a permutation $\pi_{m-2} : B_n \to B_n$ which has the desired effect on $\omega|_{G^c}$ and is the identity on $\omega|_G$. Hence $\bar{d}(\omega \circ \pi_{m-2} \circ \pi_{m-1}, \omega') < (1-a)^2$.

Repeat this process at most $(m-1)$ times. At the ith stage we will have permutations $\pi_i, \pi_{i+1}, \ldots, \pi_{m-1}$ such that $\bar{d}(\omega \circ \pi_i \circ \cdots \circ \pi_{m-1}, \omega') < (1-a)^{m-i}$ and a set G_i which is the union of $\bar{\epsilon}$-interiors of τ_{k_j} matched by π_j for $i \le j \le m-1$. If $|G_i^c|/n^d > \bar{\epsilon}$ then since (i) and (v) hold for all necessary k_i we can argue as before that (f) of the nesting lemma holds. Hence we can apply the nesting lemma to yield a permutation π_{i-1}.

Set $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_{m-1}$. To finish the proof all that remains is to show $m(\pi) < \epsilon$. Thus, according to Definition 2.1.1 we need a set $C \subset B_n$ such that $|C| > (1-\epsilon)n^d$, and for every $\vec{u}, \vec{v} \in C$, $\|\pi\vec{u} - \pi\vec{v} - (\vec{u} - \vec{v})\| < \epsilon\|\vec{u} - \vec{v}\|$.

Note that $(G_i - G_{i+1})$ is the set of $\bar{\epsilon}$-interiors of τ_{k_i} matched by π_i. Let C_i be the union of the $\sqrt{\bar{\epsilon}}$-interiors of these matched boxes and put $C = \cup_{i=1}^{m-1} C_i$. Note that C is entirely contained in $\cup_{i=1}^{m-1}(G_i - G_{i+1})$. If a_i is the exact percentage of G_{i+1}^c matched by π_i, then the percentage of B_n being removed with these $\sqrt{\bar{\epsilon}}$-collars at stage i is $2d\sqrt{\bar{\epsilon}}a_i \prod_{j>i}(1-a_j)$. Thus in total we have removed $\sum_{i=1}^{m-1} 2d\sqrt{\bar{\epsilon}}a_i \prod_{j>i}(1-a_j) < 2d\sqrt{\bar{\epsilon}}(1)$ and we have $|C|/|B_n| > 1 - \epsilon/2 - 2d\sqrt{\bar{\epsilon}} > 1 - \epsilon$.

Assume $\vec{u}, \vec{v} \in C$ lie in τ_{k_i}, τ_{k_j}, two towers that are matched by π_i and π_j respectively. If $i = j$ and $\vec{u}, \vec{v}$ lie in the exact same τ_{k_i} then $\|\pi\vec{u} - \pi\vec{v} - (\vec{u} - \vec{v})\| = 0$. If $\vec{u}, \vec{v}$ lie in different k_i-towers matched by π_i, note that $\|\vec{u} - \vec{v}\| > \sqrt{\bar{\epsilon}}s^{k_i}$ and $\|\pi\vec{u} - \pi\vec{v} - (\vec{u} - \vec{v})\| \le 2\bar{\epsilon}b^{k_i}$. Now assume $i > j$ (thus $\vec{u}$ is matched first in our construction). Then $\|\pi\vec{u} - \pi\vec{v} - (\vec{u} - \vec{v})\| \le \bar{\epsilon}b^{k_i} + b^{k_{i-1}} + \cdots + b^{k_j} + \bar{\epsilon}b^{k_j}$. We can use (i) to show this is less than $2\bar{\epsilon}b^{k_i}$.

Finally,

$$\begin{aligned} \|\pi\vec{u} - \pi\vec{v} - (\vec{u} - \vec{v})\| &\le \frac{2\bar{\epsilon}b^{k_i}}{\|\vec{u} - \vec{v}\|}\|\vec{u} - \vec{v}\| \le \frac{2\bar{\epsilon}b^{k_i}}{\sqrt{\bar{\epsilon}}s^{k_i}}\|\vec{u} - \vec{v}\| \\ &\le 2\sqrt{\bar{\epsilon}}\alpha\|\vec{u} - \vec{v}\| < \epsilon\|\vec{u} - \vec{v}\|, \end{aligned}$$

as needed. □

3. *A loosely Bernoulli $\mathbb{Z}^2$ Action with a non-loosely Bernoulli subaction*

3.1. *Background.* In [F] Feldman constructed an example of a zero entropy $\mathbb{Z}$ action which is not LB. In this section we use his example to construct a $\mathbb{Z}^2$ zero entropy LB action with a non-LB subaction.

We first remind the reader of Feldman's construction. Start with $N(1) = 200 \cdot 2^4$ symbols $\{a_{1,1}, \ldots, a_{1,N(1)}\}$, of length $L'(1) = 1$. Call this collection the 1-array. Inductively form strings of length $L'(n+1)$ by concatenating members of the n-array. The strings $a_{n,j}$ are called n-symbols. There will be $N(n+1) = 200 \cdot 2^{n+4}$ different

$(n+1)$-symbols, which are listed in the $n+1$-array:

$$\begin{aligned}
a_{n+1,1} &= ((a_{n,1})^{N(n)^2} \cdots (a_{n,N(n)})^{N(n)^2})^{N(n)^{2N(n+1)}} \\
a_{n+1,2} &= ((a_{n,1})^{N(n)^4} \cdots (a_{n,N(n)})^{N(n)^4})^{N(n)^{2(N(n+1)-1)}} \\
&\vdots \\
a_{n+1,i} &= ((a_{n,1})^{N(n)^{2i}} \cdots (a_{n,N(n)})^{N(n)^{2i}})^{N(n)^{2(N(n+1)-i+1)}} \\
&\vdots \\
a_{n+1,N(n+1)} &= ((a_{n,1})^{N(n)^{2N(n+1)}} \cdots (a_{n,N(n)})^{N(n)^{2N(n+1)}})^{N(n)^2}.
\end{aligned}$$

The sub-blocks of $a_{n+1,i}$ of length $L'(n)N(n)^{2i+1}$ consisting of repeated strings of n-symbols will be referred to as cycles and denoted by $c_{n+1,i}$. Thus $a_{n+1,i}$ consists of $c_{n+1,i}$ repeated $N(n)^{2(N(n+1)-i+1)}$ times. Note that $L'(n+1) = L'(n)N(n)^{2N(n+1)+3}$.

Let $X = [0,1]$ and μ be Lebesgue measure. Denote by (X, μ, T) the point transformation constructed in **[F]** from the symbolic construction described above. An extensive analysis of the properties of T can be found in **[F]** and **[ORW]**. We state here, without proof, the results from **[F]** pertinent to our work.

LEMMA 3.1.1. *If $N(n) > 200 \cdot 2^{n+2}$, then for all positive integers s and t and all n, if $i < j$,*

$$\bar{f}(a_{n,i}^s, a_{n,j}^t) \geq 1 - \frac{1}{8}\left(1 - \left(\frac{1}{2^n}\right)\right).$$

Recall that the proof of Lemma 3.1.1 rests on two ingredients:

1. if $i < j$ the length of a cycle in $a_{n,i}$ is exponentially smaller than the length of a repeated block in $a_{n,j}$;
2. each n-symbol appears exactly $1/N(n)$ of the time in any given $(n+1)$-symbol.

In particular, the proof is independent of the particular order the n-symbols appear in an $(n+1)$-symbol.

As a consequence of this lemma, **[F]** has the following result.

THEOREM 3.1.2. *(X, μ, T) is an ergodic, measure-preserving transformation of zero entropy which is not LB.*

3.2. *The symbolic structure of the $\mathbb{Z}^2$ action.* Our two-dimensional construction is designed so its horizontal action will mimic the periodic structure of Feldman's example. To prove that the horizontal action is not LB we will prove the analog of Lemma 3.1.1 and proceed as in **[F]**.

We now inductively define the two-dimensional symbolic blocks we will use in our construction. At each stage $n+1$, there will be one two-dimensional block labelled τ_{n+1}, which we call the $(n+1)$-tower. It will be a rectangle of width $L(n+1) = L(n)\,(N(n)F(n))^{2N(n+1)+3}$, and height

$$\overline{L}(n+1) = \left[\frac{L(n+1)}{F(n+1)N(n+1)}\right] F(n+1)N(n+1),$$

where $L(2) = L'(2)$, $F(2) = 1$, $N(n)$ is as above and $F(n+1) = \overline{L}(n) + F(n)N(n) - 1$. Here, for $x \in \mathbb{R}$, $[x]$ denotes the integer part of x.

The symbolic structure of the rows of τ_{n+1} will be of special interest; we label the ith row of the $(n+1)$-tower by $r_{n+1,i}$. Our convention will be that $r_{n+1,1}$ is the bottom row in τ_{n+1}. The symbols used in the towers will be the symbols in Feldman's 1-array and the symbol $*$. The symbol $*$ will be called a spacer, and any row containing a spacer will be called a spacer row.

We start by describing τ_2, a rectangle of width $L(2)$ and height $\overline{L}(2)$. It will consist of copies of a block B_2 stacked vertically on top of each other, where B_2 consists of $N(2)$ distinct sub-blocks, $\hat{C}_{2,j}$, and $\hat{C}_{2,j}$ is constructed from horizontal repetitions of the cycle $c_{2,j}$. More specifically, $\hat{C}_{2,j}$ equals $c_{2,j}^{N(1)^{2(N(2)-j+1)}}$ and B_2 is the block of symbols obtained by stacking these sub-blocks on top of each other in ascending order of j. Then τ_2 is constructed by stacking $L(2)/N(2)$ copies of B_2 on top of each other. We then have, for each $0 < i \leq \overline{L}(2)$ with $kN(2) < i \leq (k+1)N(2)$ for some $k \geq 0$,

$$r_{2,i} = a_{2,i-kN(2)}.$$

For the inductive step, suppose τ_n is given. The next tower, τ_{n+1}, will be a rectangle of width $L(n+1)$ and height $\overline{L}(n+1)$ and again will consist of copies of a block B_{n+1} stacked vertically, where B_{n+1} consists of $N(n+1)$ distinct sub-blocks, $\hat{C}_{n+1,j}$. A sub-block again is constructed from horizontal repetitions of a cycle, now denoted $C_{n+1,j}$. $C_{n+1,j}$ will be a rectangular block of symbols of width $L(n)(N(n)F(n))^{2j+1}$ and height $F(n+1) = \overline{L}(n) + N(n)F(n) - 1$. We first describe how to construct τ_{n+1} using the $C_{n+1,j}$, then describe the cycles themselves.

Define $\hat{C}_{n+1,j}$ to be $(C_{n+1,j})^{(F(n)N(n))^{2N(n+1)-2j+2}}$. Then B_{n+1} is the block of symbols obtained by stacking $\hat{C}_{n+1,j}$ on top of each other in ascending order of j (see Figure 1). Then τ_{n+1} is constructed by stacking $[L(n+1)/F(n+1)]$ copies of B_{n+1} on top of each other. Note that this number is greater than $(N(n)F(n))^{2N(n+1)+2}$.

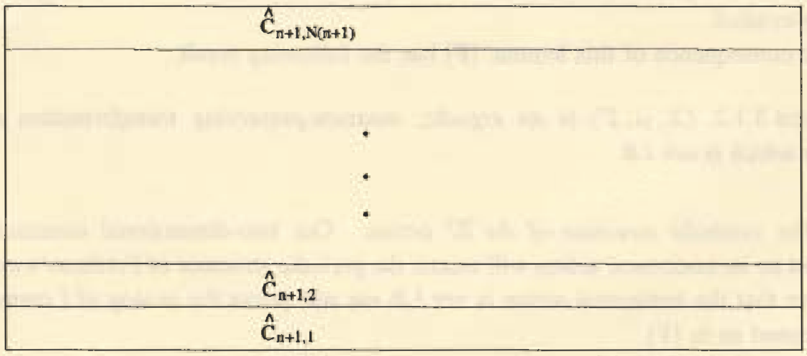

FIGURE 1. B_{n+1}

Now for the $C_{n+1,j}$ themselves. The cycle $C_{n+1,j}$ will consist of spacers and $(F(n)N(n))^{2j+1}$ copies of τ_n. Notate the entries in this rectangle as one would a matrix so (i, j) denotes the symbol in the ith row, jth column, except that our convention is still that the bottom row is row number 1. The copies of τ_n will be placed in groups of $(F(n)N(n))^{2j}$, where the copies within a grouping are placed on the same row. The first grouping will be placed at $(F(n)N(n), 1)$ and the second grouping at $(F(n)N(n) - 1, L(n)(F(n)N(n))^{2j})$ etc, until the last grouping starts at coordinate $(1, (L(n)(F(n)N(n))^{2j})(F(n)N(n) - 1)))$. The remaining coordinates will be filled with the symbol $*$ (see Figure 2).

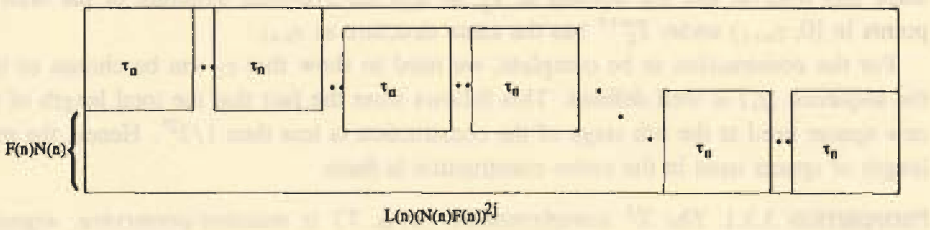

FIGURE 2. $C_{n+1,j}$

Denote the ith row in $C_{n+1,j}$ by $C_{n+1,j,i}$. Then for $1 \le i \le F(n)N(n) - 1$

$$C_{n+1,j,i} = (*)^{L(n)(F(n)N(n)^{2j})(F(n)N(n)-i)} r_{n,1}^{(F(n)N(n))^{2j}} r_{n,2}^{(F(n)N(n))^{2j}} \cdots r_{n,i}^{(F(n)N(n))^{2j}}.$$

We will refer to rows such as these, which consist of spacers along with copies of rows from τ_n, as new spacer rows. After the $(F(n)N(n) - 1)$th row, the new spacer rows end and do not start again until the $(F(n+1) - \overline{L}(n))$th row. In between the rows are of the form

$$r_{n,i}^{(F(n)N(n))^{2j}} r_{n,i+1}^{(F(n)N(n))^{2j}} \cdots r_{n,i+F(n)N(n)-1}^{(F(n)N(n))^{2j}}.$$

When the new spacer rows start again, this time the spacers appear in increasing amounts starting at the end of the rows. If we are at the ith row from the top with $0 \le i \le F(n)N(n) - 1$, then the row is of the form

$$r_{n,\overline{L}(n)-i}^{(F(n)N(n))^{2j}} r_{n,\overline{L}(n)-i+1}^{(F(n)N(n))^{2j}} \cdots r_{n,\overline{L}(n)}^{(F(n)N(n))^{2j}} (*)^{(L(n)(F(n)N(n))^{2j}-(F(n)N(n)-i))}.$$

As in Feldman's example, there are $N(n + 1)$ cycles, each with a distinct periodic structure. A row from $\hat{C}_{n+1,i}$ will be called an $(n + 1)$-row of type i. The periodic structure of rows of different types are related to each other in the same way as different $(n + 1)$-names in Feldman's construction. There now are, however, many different rows of the same type. Each $\hat{C}_{n+1,i}$ has $F(n + 1)$ rows, each a distinct $L(n + 1)$ name. We thus have $F(n+1)N(n+1)$ distinct horizontal $L(n+1)$-names in τ_{n+1}, but these names divide into $N(n + 1)$ distinct types. Notice that each name appears $[L(n + 1)/F(n + 1)]$ times in τ_{n+1}.

3.3. *Cutting and stacking.* We now wish to find a process (T, P) whose names have the symbolic structure described in the previous section. We will indicate a cutting and stacking construction which will yield a subset $I \subset [0, 1]$ of full measure, a partition P of I, and a $\mathbb{Z}^2$ action T so the process (T, P) is as desired.

Set $I_1 = [0, z_1)$ with $0 < z_1 < 1$ to be defined later. We first divide I_1 into subintervals and label these by the symbols $a_{1,p}$, $1 \le p \le N(1)$ and define $T_{\vec{v}}^1$ so that the orbit of these points have the same symbolic structure as τ_2. The interval $[z_1, 1]$ will serve as our spacer interval and will be labeled with the symbol $*$.

Suppose $T_{\vec{v}}^n$ is defined on $[0, z_n)$. Then at the next stage of the construction we will cut an appropriate size interval from $[z_n, 1]$ to use as the new spacers. We then cut and stack this interval and the domain of $T_{\vec{v}}^n$ so that the symbolic structure of the orbit of points in $[0, z_{n+1})$ under $T_{\vec{v}}^{n+1}$ has the same structure as τ_{n+1}.

For the construction to be complete, we need to show that z_1 can be chosen so that the sequence $\{z_i\}$ is well defined. This follows from the fact that the total length of the new spacer used at the nth stage of the construction is less than $1/2^{2^n}$. Hence, the total length of spacer used in the entire construction is finite.

PROPOSITION 3.3.1. *The $\mathbb{Z}^2$ transformation (X, μ, T) is measure-preserving, ergodic, zero entropy and loosely Bernoulli.*

Proof. That T is measure preserving follows easily from the construction of the $T_{\vec{e}_i,n}$. Note also that $L(n)/\overline{L}(n) \ge 1 - 1/F(n)N(n)$, so in particular Definition 2.2.2 is satisfied with $\alpha = 2$. T is then square rank one, and the remaining facts follow from **[Fr]** and **[PR]**. □

3.4. *The horizontal subaction.* We now turn our attention to the horizontal action $(X, \mu, T_{\vec{e}_1})$. To ease notation we refer to new spacer rows in a tower as bad rows, and to non-new spacer rows as good rows.

PROPOSITION 3.4.1. *The $\mathbb{Z}$ action $(X, \mu, T_{\vec{e}_1})$ is measure preserving, ergodic, and has zero entropy.*

Proof. The cutting and stacking construction of T guarantees that $T_{\vec{e}_1}$ is measure preserving.

Ergodicity follows easily from the geometry of the cutting and stacking construction. The key facts are that for all k and n, each good row in τ_{n+k} sees all the rows from τ_n, and the good rows of τ_{n+k} are a proportion greater than $1 - 1/100 \cdot 2^{n+6}$ of the rows in τ_{n+k}.

To complete the proof, note that the total number of strings of length $L(n)$ is at most $L(n)(L(n) + 1)^2$. □

Our construction was arranged so that names of different types have vastly differing periodic structures. We now need a result analogous to Lemma 3.1.1 to show that they are $\overline{f}$ far from one another. In particular, the rows in τ_3 which are not new spacer rows consist of either Feldman names or permutations of Feldman names. We have made the observation that the proof of Lemma 3.1.1 does not depend on the particular order that the sub-blocks appear in Feldman's construction, so Lemma 3.1.1 will still hold, and two such rows of different type from τ_3 will match at best $\frac{1}{8}(1 - 1/2^3)$. New spacer rows from τ_3, however, are a different story. Two such rows could match exceedingly well. These rows, however, are a small proportion of the rows in τ_3, and hence do not affect the induction needed to prove a non-matching lemma.

LEMMA 3.4.2. *Let $r_{n,l}$ and $r_{n,l'}$ be good rows from τ_n which are of different type. Then for all positive integers s and t,*

$$\overline{f}(r_{n,l}^s, r_{n,l'}^t) \geq 1 - \frac{1}{8}\left(1 - \frac{1}{2^n}\right).$$

Proof. We will essentially repeat the argument in [F] with modifications made for the appearances of spacers in our construction.

Recall from the construction that two rows are of different type if and only if they come from different cycles. We prove the result by induction. As observed earlier, a straightforward application of the proof of Lemma 3.1.1 will show that the result is true for the case $n = 3$.

Now suppose that the result is true for some n, and let $r_{n+1,l}$ and $r_{n+1,l'}$ be as in the statement of the lemma. Suppose further that $r_{n+1,l}$ comes from $\hat{C}_{n+1,i}$ and $r_{n+1,l'}$ comes from $\hat{C}_{n+1,j}$ with $j = i + k$, k some positive integer. Then $r_{n+1,l}$ is of the form

$$(r_{n,v}^{(N(n)F(n))^{2i}} r_{n,v+1}^{(N(n)F(n))^{2i}} \cdots r_{n,v+F(n)N(n)-1}^{(N(n)F(n))^{2i}})^{t(F(n)N(n))^{2N(n+1)-2i+2}}.$$

Let $\alpha_v = r_{n,v}^{(F(n)N(n))^{2i}}$. The row can then be written as

$$(\alpha_v\, \alpha_{v+1} \cdots \alpha_{v+F(n)N(n)-1})^{t(F(n)N(n))^{2N(n+1)-2i+2}},$$

and the row $r_{n+1,l'}$ is then of the form

$$(\alpha_w^{(F(n)N(n))^{2k}}\, \alpha_{w+1}^{(F(n)N(n))^{2k}} \cdots \alpha_{w+F(n)N(n)-1}^{(F(n)N(n))^{2k}})^{s(F(n)N(n))^{2N(n+1)-2j+2}}.$$

The significant facts to note here are that these strings each see all the distinct rows from τ_n and each row occurs exactly once. Further, the periodic structures of the two strings are different.

As in [F] we observe that each sub-block $\alpha_h^{(F(n)N(n))^{2k}}$ of the second row must be matched with a substring of the first row of the form

$$(b(\alpha_v\, \alpha_{v+1} \cdots \alpha_{v+F(n)N(n)-1})^q c),$$

where b and c are end and beginning blocks of the repeated string. We compare instead the strings $\alpha_h^{(F(n)N(n))^{2k}}$ with the completed strings

$$x = (\alpha_v\, \alpha_{v+1} \cdots \alpha_{v+F(n)N(n)})^{q+2},$$

and argue as in [F] that the match may be decreased from the original but by a factor less than $\theta = (1 - 1/100 \cdot 2^{n+1})$.

Now, because of the relative lengths of $\alpha_h^{(F(n)N(n))^{2k}}$ and x each α_p in x must be matched with a substring of $\alpha_h^{(F(n)N(n))^{2k}}$. These are of the form $a'(\alpha_h)^u b'$, where a' and b' are end and beginning blocks respectively of α_h. We argue as before that if we compare α_p with $(\alpha_h)^{u+2}$ we will decrease the match by a factor no more than θ.

We are now comparing the two strings $\alpha_p = r_{n,p}^{(F(n)N(n))^{2i}}$ and $\alpha_h^{u+2} = r_{n,h}^{(F(n)N(n))^{2i}(u+2)}$. If both $r_{n,p}$ and $r_{n,h}$ are good rows from τ_n then we can apply the induction hypothesis. If not, then they could match well.

We can now specify the match between $r'_{n+1,l}$ and $r^{s}_{n+1,l'}$ with the two modifications made of completing strings. Let g_n denote the proportion of good rows in τ_n and $b_n = 1 - g_n$. Then g_n proportion of the time, α_h will be constructed from a good row. Also, g_n proportion of the time, $r_{n,p}$ will be a good row. In this case $1/N(n)$ of the time we will have two rows of the same type and they will match well. The rest of the time we can apply the inductive hypothesis. The remaining b_n percentage of the time, $r_{n,p}$ will be a bad row and we will assume that the strings can match arbitrarily well. Finally, b_n percentage of the time $r_{n,h}$ itself is a bad row where again we estimate by a perfect match.

The final match then is:

$$g_n\left[g_n\left(\left(\frac{N(n)-1}{N(n)}\right)\cdot\frac{1}{8}\left(1-\frac{1}{2^n}\right)+\frac{1}{N(n)}\cdot 1\right)+b_n\cdot 1\right)\frac{1}{\theta^2}\right]+b_n\cdot 1$$
$$<\frac{1}{\theta^2}\left(\frac{1}{8}\left(1-\frac{1}{2^n}\right)+\frac{1}{N(n)}+2b_n\theta^2\right)<\frac{1}{8}\left(1-\frac{1}{2^{n+1}}\right). \qquad \square$$

THEOREM 3.4.3. *The one-dimensional subgroup actions of a zero entropy, LB $\mathbb{Z}^2$ action are not all necessarily LB.*

Proof. Consider the $\mathbb{Z}^2$ action (X, μ, T) construction in this section. By Propositions 3.3.1 and 3.4.1 we have that T and $T_{\vec{e}_1}$ satisfy the hypotheses above.

To argue that $(X, \mu, T_{\vec{e}_1})$ is not LB we let P be the time zero partition into the symbols $\{1, \dots, N(1)\}$ and slightly modify the argument in [F] to adjust for the spacers in our construction. As in [F], Lemma 3.4.2 guarantees that if the $L(n)$-names of two points y_1, y_2 are constructed from good rows in τ_n which are of different type, then the $L(n)$-names of y_1 and y_2 match no better than $\frac{1}{4}$. The bad rows are a very small proportion of the rows in τ_n, hence in any set of large measure, there exist points y_1 and y_2 as described above. $\square$

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