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Factoring higher-dimensional shifts of finite type onto the full shift

AIMEE JOHNSON† and KATHLEEN MADDEN‡

† *Department of Mathematics and Statistics, Swarthmore College, Swarthmore, PA 19081, USA*
(e-mail: aimee@swarthmore.edu)
‡ *Department of Mathematics and Computer Science, Drew University, Madison, NJ 07940, USA*
(e-mail: kmadden@drew.edu)

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Abstract. A one-dimensional shift of finite type $(X, \mathbb{Z})$ with entropy at least $\log n$ factors onto the full n-shift. The factor map is constructed by exploiting the fact that X, or a subshift of X, is conjugate to a shift of finite type in which every symbol can be followed by at least n symbols. We will investigate analogous statements for higher-dimensional shifts of finite type. We will also show that for a certain class of mixing higher-dimensional shifts of finite type, sufficient entropy implies that $(X, \mathbb{Z}^d)$ is finitely equivalent to a shift of finite type that maps onto the full n-shift.

1. *Introduction*

An important question in the study of dynamical systems is the question of when one system can occur as the factor of another. For one-dimensional shifts of finite type there are a variety of results addressing this issue. For example, the full shift on n symbols is in some sense the simplest of the one-dimensional shifts of finite type with entropy at least $\log n$ since it is a factor of each of them [**LM**, Ch. 5]. In this work, we will give a sufficient condition for a higher-dimensional shift of finite type to factor onto the full n-shift; however, it remains an open question whether entropy alone is sufficient in higher dimensions.

In the literature, extending one-dimensional results to higher dimensions has often required additional mixing assumptions. For example, Meester and Steif [**MS**] extend the well-known, one-dimensional characterization of entropy preserving maps [**LM**, Theorem 8.1.16] under the assumption of strong irreducibility. Lightwood [**L**] extends Krieger's [**Kr**] topological universal model results to higher-dimensional mixing, square filling shifts of finite type. Robinson and Şahin [**RS**] extend Krieger's [**Kr**] measurable universal model results to higher-dimensional shifts of finite type satisfying the uniform filling property. Strong mixing conditions may be necessary in higher dimensions; in particular, it is known that Robinson and Şahin's measure-theoretic result does not hold under the assumption of topological mixing alone [**QS**].

We will introduce a mixing condition called corner gluing which is stronger than topological mixing but which is implied by each of the mixing properties mentioned in the previous paragraph. We will show in Theorem 4.1 that when a higher-dimensional shift of finite type with sufficient entropy is corner gluing, then it is the finite-to-one factor of a shift of finite type that factors onto the full n-shift.

2. *Background*

Let $\mathcal{A} = \{1, 2, \ldots, n\}$ be a finite alphabet and consider the compact metric space $X^d_{[n]} = \mathcal{A}^{\mathbb{Z}^d}$. For each $\vec{v} \in \mathbb{Z}^d$, let $x_{\vec{v}}$ denote the symbol in position $\vec{v}$ in array $x \in X^d_{[n]}$. Let

$$\sigma_d : X^d_{[n]} \times \mathbb{Z}^d \to X^d_{[n]}$$

be the continuous $\mathbb{Z}^d$-action defined by

$$(\sigma_d(x, \vec{v}))_{\vec{w}} = x_{\vec{v}+\vec{w}}$$

for all $\vec{v}, \vec{w} \in \mathbb{Z}^d$ and all $x \in X^d_{[n]}$. We call σ_d the *d-dimensional shift map* and $(X^d_{[n]}, \sigma_d)$ the *d-dimensional full n-shift*. When it causes no confusion, we will denote the d-dimensional full n-shift by $(X_{[n]}, \sigma_d)$.

If X is a closed, shift invariant subspace of $X^d_{[n]}$, we call (X, σ_d) a d-dimensional subshift or shift space.

For $x \in X$ and $B \subset \mathbb{Z}^d$, we will denote the configuration of symbols appearing in x in the locations determined by B as x_B. We define $S(X, B) = \{x_B : x \in X\}$. Thus $S(X, B)$ is all configurations occurring in the locations determined by B in any $x \in X$. We also let

$$B_m = \{(a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d : |a_i| \leq m \text{ for } 1 \leq i \leq d\}$$

and for $\vec{k} \in \mathbb{N}^d$,

$$D_{\vec{k}} = \{(a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d : 0 \leq a_i < k_i \text{ for } 1 \leq i \leq d\}.$$

We will let $D_m = D_{\vec{k}}$ when $k_i = m$ for all $1 \leq i \leq d$.

Given a shift space (X, σ_d) and an integer $M > 0$, we define the *Mth higher power shift space* X^M by 'chopping up' arrays in X into M^d blocks. That is, the alphabet of X^M is $S(X, D_M)$ and we let $\gamma : X \to X^M$ be defined by $\gamma(x)_{\vec{v}} = x_{D_M + M\vec{v}}$ for all $\vec{v} \in \mathbb{Z}^d$. The *Mth higher power shift map*, σ^M_d, is the map defined by

$$(\sigma^M_d(\gamma(x), \vec{v}))_{\vec{w}} = \gamma(x)_{\vec{v}+\vec{w}} = x_{D_M + M(\vec{v}+\vec{w})}$$

for all $\vec{v}, \vec{w} \in \mathbb{Z}^d$ and all $x \in X$.

A *d-dimensional shift of finite type* is a subshift (X, σ_d) defined by a list of allowable configurations on B_m for some $m > 0$. We will call a configuration of symbols on an arbitrary set $B \subset \mathbb{Z}^d$ *allowable* if all configurations on subsets $B_m + \vec{v} \subseteq B$ are allowable.

By moving to a higher block presentation if necessary, a d-dimensional shift of finite type can be described by adjacency rules given by d zero-one adjacency matrices $A_1, A_2, \ldots, A_d$; symbol α may appear next to symbol β in the e_i direction if and only if $A_i(\alpha, \beta) = 1$.

A *block map* $\phi : X \to Y$ between shift spaces (X, σ_d) and (Y, σ_d) is defined by a mapping Φ between $S(X, B_m)$ for some m and the symbols occurring in Y. If $\Phi : S(X, B_m) \to \mathcal{A}$ where $\mathcal{A}$ is the alphabet for Y, then $\phi(x)_{\vec{v}} = \Phi(x_{\vec{v}+B_m})$ for all $\vec{v} \in \mathbb{Z}^d$. It is easily verified that maps between shift spaces are continuous and commute with the shift map if and only if they are defined in this way. If a block map is onto, it is called a *factor map* and we say that X factors onto Y. If a factor map is one-to-one, it is called a *conjugacy* and conjugate shift spaces exhibit identical dynamical properties.

Difficulties arise in higher dimensions which do not occur in the traditional one-dimensional case. For example, given a single adjacency matrix, it is relatively easy to determine whether the corresponding one-dimensional shift of finite type is non-empty. In higher dimensions, the question of whether there are *any* arrays of symbols satisfying the adjacency rules given by the adjacency matrices is referred to as the *non-emptiness problem* and in general this problem is undecidable **[B, R]**. However, the hypotheses of the theorems in this paper will require that shift spaces have additional properties (either a corner condition or corner gluing as defined in the next section) and these properties imply that the shift spaces are non-empty, provided the one-dimensional horizontal and vertical shift spaces are non-empty.

Extensive background material on one-dimensional shifts of finite type can be found in **[LM]** or **[K]**. Lind and Marcus also provide a good overview of higher-dimensional shifts in **[LM**, Ch. 3]. A paper by Quas and Trow contains many interesting general results about higher-dimensional shifts of finite type and their subshifts **[QT]**.

3. *Factoring onto the full shift and the corner condition*

We next give several definitions which will lead in Theorem 3.2 to a sufficient condition for a shift of finite type of any dimension to factor onto the full n-shift.

Let $\vec{c} = (1, 1, \ldots, 1) \in \mathbb{Z}^d$. A *$d$-dimensional corner C* is the subset of $\mathbb{Z}^d$ given by

$$\{\vec{a} = (a_1, a_2, \ldots, a_d) : a_i \in \{0, 1\} \text{ for all } 1 \le i \le d \text{ and } \vec{a} \ne \vec{c}\}.$$

We will call $\vec{c}$ the *corner position.* A *corner configuration $\mathcal{C}$* is an element of $S(X, C)$.

Definition 3.1. A shift of finite type (X, σ_d) has corner condition n if, for each corner configuration, there are at least n allowable choices for the corner position.

Note that, in the case of $d = 1$, the set of corner configurations is the set of symbols in the alphabet $\mathcal{A}$, and the corner condition implies that each symbol has at least n allowable followers.

If a shift of finite type has corner condition n, then it factors onto the full n-shift. This fact was shown to us by Paul Trow.

THEOREM 3.2. **[T]** *For any $d \ge 1$, if a shift of finite type (X, σ_d) has corner condition n, then (X, σ_d) factors onto $(X^d_{[n]}, \sigma_d)$.*

Proof. Each corner configuration C has at least n choices for the corner position $\vec{c}$; partition these choices into n non-empty sets $C_1, C_2, \ldots, C_n$. We can define a map

$$\Phi : S(X, B_1) \to \{1, 2, 3, \ldots, n\}$$

via $\Phi(x_{B_1}) = i$ if and only if $x_{\vec{0}} \in C_i$ for corner configuration C given by $x_{C-\vec{c}}$. The map Φ determines a block map $\phi : X \to X^d_{[n]}$. We need to show that ϕ is an onto mapping.

Let y be any point in $X^d_{[n]}$ and $k \in \mathbb{N}$. We will show that there is a point $x \in X$ with $\phi(x)_{B_k} = y_{B_k}$. A standard compactness argument then implies that ϕ is onto.

To construct x, start with an arbitrary $\overline{x} \in X$ and consider $\overline{x}_D$ where D is

$$D = \bigcup_{1\leq j\leq d} \{(i_1, \ldots, i_d) : i_j = -k - 1 \text{ and for all other } n, -k-1 \leq i_n \leq k+1\}.$$

(For $d = 1$, D is the position just to the left of B_k. For $d = 2$, D is the left and bottom border of B_k, and so on.)

Now consider the corner configuration C found at the translated corner $C - (k + 1)\vec{c}$. There are at least n allowable choices for the symbol in the translated corner position $-k\vec{c}$; if $y_{-k\vec{c}} = i$, choose a symbol from C_i to put in position $-k\vec{c}$. Continue in this way for all translated corner positions in B_k giving an allowable configuration in X. Extend this to give the desired $x \in X$. □

In one dimension, the corner condition is one of several equivalent properties of an irreducible shift of finite type:

THEOREM 3.3. [**LM**, Theorem 5.5.6] *If (X, σ_1) is an irreducible one-dimensional shift of finite type, then the following are equivalent:*

(1) *a subshift of (X, σ_1) is conjugate to a one-dimensional shift of finite type (Y, σ_1) which has corner condition n;*
(2) *(X, σ_1) maps onto the full shift;*
(3) *$h(X) \geq \log n$.*

For higher-dimensional shifts of finite type, (1) implies (2) which implies (3), but it is unknown whether the converse of these statements holds. In our main theorem, Theorem 4.1, we will prove a partial converse. This theorem will require an additional mixing condition, called 'corner gluing', and we conclude this section with its definition.

Definition 3.4. A shift of finite type (X, σ_d) is corner gluing if there exists $l > 0$ such that, given any two finite subsets $E_1, E_2 \in \mathbb{Z}^d$ as defined below (and as illustrated in Figure 1 for $d = 2$) and allowable configurations $\mathcal{R}_1$ and $\mathcal{R}_2$ on these subsets, there exists a point $x \in X$ with $x_{E_1} = \mathcal{R}_1$ and $x_{E_2} = \mathcal{R}_2$.

$E_1 = D_{\vec{k}} + l\vec{c}$ for some $\vec{k} \in \mathbb{N}^d$ and $E_2 = (D_{\vec{k'}} - (\vec{k'} - \vec{k} - l\vec{c}))\backslash D_{(\vec{k}+l\vec{c})}$ for some $\vec{k'} \in \mathbb{N}^d$ with $k'_i > k_i + l$ for $1 \leq i \leq d$.

In one dimension, the corner gluing property is equivalent to topological mixing. In higher dimensions, the corner gluing property is stronger than topological mixing but is implied by strong irreducibility, the uniform filling property or the square filling, mixing property.

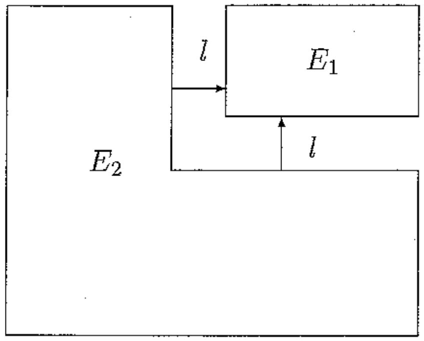

FIGURE 1. Subsets E_1 and E_2 from the definition of corner gluing.

In the next section, we will see that, with the assumption of corner gluing, a shift of finite type with sufficient entropy is finitely equivalent to a shift of finite type that maps onto the full shift.

4. *Higher powers and factoring onto the full shift*

THEOREM 4.1. *Suppose that (X, σ_d) is corner gluing and $h(X) > \log(n)$. Then (X, σ_d) is the finite-to-one factor of a shift of finite type that maps onto the full shift.*

Theorem 4.1 is an immediate corollary of Theorems 4.2 and 4.3. Theorem 4.2 shows that a corner gluing shift space with sufficient entropy has a higher power with a corner condition. In Theorem 4.3 this higher power is used to construct the finite extension of the original shift space which maps onto the full shift.

THEOREM 4.2. *Suppose that (X, σ_d) is corner gluing with constant $l > 0$. Then $h(X) > \log n$ if and only if given $c > 0$, for sufficiently large M, the higher power (X^M, σ_d^M) of (X, σ_d) has corner condition n^{M^d+c}.*

Proof. If (X, σ_d) has a higher power (X^M, σ_d^M) with corner condition n^{M^d+c}, it is an easy exercise to check that $h(X) > \log n$.

We next suppose that $h(X) > \log n$. Choose any $c > 0$. Since

$$h(X) = \lim_{m\to\infty} \frac{1}{m^d} \log |S(X, D_m)| > \log n$$

it is clear that

$$\lim_{m\to\infty} \frac{1}{(m+l)^d + c} \log |S(X, D_m)| > \log n.$$

Thus we can choose m large enough so that

$$\frac{1}{(m+l)^d + c} \log |S(X, D_m)| > \log n,$$

and $|S(X, D_m)|$, the number of configurations occurring in X on D_m, is strictly greater than $n^{(m+l)^d+c}$.

Let $M = m + l$. Let C be any corner configuration in X^M. So $C = x_F$ for some $x \in X$ where $F = \bigcup_{\vec{v} \in C} D_M + M\vec{v}$. We must show that there are at least n^{M^d+c} choices for the configuration occurring on $D_M + M\vec{c}$. We can put any of the more than $n^{(m+l)^d+c}$ configurations from $S(X, D_m)$ in positions $D_m + (M+l)\vec{c}$ and extend via the corner gluing property to obtain an allowable configuration on $F \cup (D_M + M\vec{c})$. Thus there are at least n^{M^d+c} choices for the corner position of C, as desired. □

We point out that entropy greater than $\log n$ is necessary. If $h(X) = \log n$, it may be the case that higher powers (X^M, σ_d^M) do not have corner condition n^{M^d} for any M. For example, consider (X, σ_1) given by the adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easily verified that A^5 consists of positive entries, and thus (X, σ_1) is corner gluing. Also the Perron–Frobenius eigenvalue for A is $\lambda = 2$, and thus $h(X) = \log 2$. However, (X^M, σ_1^M) does not have corner condition 2^M for any $M \in \mathbb{N}$ since any M block ending in 5 will have fewer than 2^M followers. To see this, it is easily verified by induction that the value for $A^M(5, j)$ is less than $3 \cdot 2^{M-4}$ for all $M \geq 4$ and $1 \leq j \leq 5$, and thus

$$\sum_{j=1}^{5} A^M(5, j) < 5(3 \cdot 2^{M-4}) < 2^M.$$

When a higher power (X^M, σ_d^M) satisfies corner condition n^{M^d}, then (X, σ_d) is finitely equivalent to a shift of finite type that maps onto the full shift as we will show in the next theorem.

THEOREM 4.3. *If (X, σ_d) has a higher power (X^M, σ_d^M) with corner condition n^{M^d} then:*
(1) *(X^M, σ_d^M) factors onto $((X_{[n]}^d)^M, \sigma_d^M)$;*
(2) *(X, σ_d) is the finite-to-one factor of a shift of finite type $(\bar{X}, \sigma_d)$ which maps onto $(X_{[n]}^d, \sigma_d)$.*

Proof. Claim (1) is an immediate corollary of Theorem 3.2 since $((X_{[n]}^d)^M, \sigma_d^M)$ is conjugate to $(X_{[n^{M^d}]}^d, \sigma_d)$.

We turn our attention to Claim (2). Notice that the corner condition on (X^M, σ_d^M) implies that $h(X) \geq \log n$ and thus, for $d = 1$, Claim (2) holds via Theorem 3.3. We will restrict our attention to the case of $d = 2$; the argument for higher dimensions is similar. Let $\mathcal{A}$ denote the symbols occurring in elements of X and let the horizontal and vertical adjacency rules be given by matrices A_h and A_v respectively. We define a new symbol set by adding sub- and superscripted symbols from $\mathcal{A}$ as follows:

$$\bar{\mathcal{A}} = \mathcal{A} \cup \{s_i, s^i, s_c : s \in \mathcal{A}, 1 \leq i \leq M - 1\}.$$

We will think of arrays in $\bar{X}$ as arrays in X with an overlaid M^2 grid given by the scripted symbols. The subscripted symbols give horizontal grid lines, the superscripted symbols

$$\mathcal{C} = \begin{array}{lllll} s_c & & & & \\ s^{M-1} & & & & \\ \vdots & & & & \\ s^2 & & & & \\ s^1 & & & & \\ s_c & s_1\, s_2 & \ldots & s_{M-1} & s_c \end{array}$$

FIGURE 2. A configuration on subset D.

give vertical grid lines and the symbols s_c occur at the intersection of horizontal and vertical grid lines. We define adjacency matrices $\bar{A}_h$ and $\bar{A}_v$ accordingly:

(1) $\bar{A}_h(s, t^i)$, $\bar{A}_h(s^i, t)$ and $\bar{A}_h(s, t)$ equal one if and only if $A_h(s, t) = 1$;
(2) for $1 \leq i \leq M - 2$, $\bar{A}_h(s_i, t_{i+1})$ equals one if and only if $A_h(s, t) = 1$;
(3) $\bar{A}_h(s_c, t_1)$ and $\bar{A}_h(s_{M-1}, t_c)$ equal one if and only if $A_h(s, t) = 1$;
(4) $\bar{A}_v(s, t_i)$, $\bar{A}_v(s_i, t)$ and $\bar{A}_v(s, t)$ equal one if and only if $A_v(s, t) = 1$;
(5) for $1 \leq i \leq M - 2$, $\bar{A}_v(s^i, t^{i+1})$ equals one if and only if $A_v(s, t) = 1$;
(6) $\bar{A}_v(s_c, t^1)$ and $\bar{A}_v(s^{M-1}, t_c)$ equal one if and only if $A_v(s, t) = 1$.

Any adjacencies not mentioned above are not allowed. Let $(\bar{X}, \sigma_2)$ denote the shift of finite type defined with adjacency rules $\bar{A}_h$ and $\bar{A}_v$ on symbols $\bar{\mathcal{A}}$.

Let $\phi : \bar{X} \twoheadrightarrow X$ be the factor map defined by the block map $\Phi : \bar{\mathcal{A}} \to \mathcal{A}$ which drops the subscripts. Clearly ϕ is M^2-to-one since, for each array $x \in X$, there are exactly M^2 ways to overlay the M^2 grid given by the sub- and superscripted symbols.

We next need to show that $\bar{X}$ factors onto $X_{[n]}$. We will need to construct a map $\psi : \bar{X} \to X_{[n]}$. To do this, first order the n^{M^2} configurations occurring in $S(X_{[n]}, D_M)$ lexicographically and denote them $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{n^{M^2}}$. Intuitively, because the sub- and superscripted symbols divide arrays in $\bar{X}$ into M^2 blocks, we will be able to define ψ by mapping the ith M^2 blocks in $x \in \bar{X}$ to $\mathcal{F}_i$. In what follows, we formalize this idea.

Let $D = \{(0, a_2)\} \cup \{(a_1, 0)\}$ where $0 \leq a_1, a_2 \leq M$, and consider the subset of $S(\bar{X}, D)$ consisting of configurations $\mathcal{C}$ on D of the form illustrated in Figure 2. (Note that each s can be any of the symbols in $\mathcal{A}$ which satisfy the adjacency rules. For simplicity, we do not indicate that in our notation.)

We may view D as a subset of

$$(D_M \cup (D_M + (M, 0)) \cup (D_M + (0, M))) - \vec{v},$$

where $\vec{v} = (M - 1)\vec{c}$, as illustrated in Figure 3.

Thus, because (X^M, σ_2^M) has corner condition n^{M^2}, given a configuration $\mathcal{C}$ as in Figure 2, there are at least n^{M^2} choices satisfying the defining adjacency rules for extending $\mathcal{C}$ to an allowable $(M + 1)^2$ configuration in $S(\bar{X}, D_{(M+1)})$. Any configuration used to extend $\mathcal{C}$ in this way will have the form illustrated in Figure 4; we will refer to such a set as a follower of $\mathcal{C}$. (Again note that our notation does not indicate that each s can be any of the symbols in $\mathcal{A}$.)

For each configuration $\mathcal{C}$ of the form shown in Figure 2, partition the choices for the followers of $\mathcal{C}$ into n^{M^2} sets denoted $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{n^{M^2}}$.

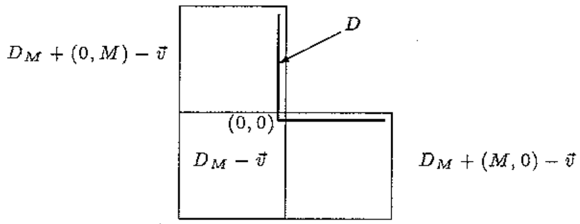

FIGURE 3. D as a subset of $(D_M \cup (D_M + (M, 0)) \cup (D_M + (0, M))) - \vec{v}$.

$$
D \quad
\begin{array}{|cccccc}
s_1 & s_2 & s_3 & \cdots & s_{M-1} & s_c \\
s & s & s & \ldots & s & s^{M-1} \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
s & s & s & \cdots & s & s^2 \\
s & s & s & \ldots & s & s^1 \\
\hline
\end{array}
$$

FIGURE 4. Extending configuration C on subset D.

We are now ready to define $\psi : \bar{X} \to X_{[n]}$. Let $x \in \bar{X}$. For each $\vec{a} \in \mathbb{Z}^2$ there exists unique $\vec{v}$, $\vec{w} \in \mathbb{Z}^2$ so that $x_{\vec{v}} = s_c$ for some $s \in \mathcal{A}$, $1 \leq w_1, w_2 \leq M$, and $\vec{a} = \vec{v} + \vec{w}$. That is, $x_{D+\vec{v}}$ is a configuration of the type shown in Figure 2 and $x_{D_M+(v_1+1,v_2+1)}$ is a configuration of the type shown in Figure 4. For $x_{D+\vec{v}} = \mathcal{C}$, $x_{D_M+(v_1+1,v_2+1)} \in \mathcal{C}_i$ for some $1 \leq i \leq n^{M^2}$. Define $\psi(x)_{\vec{a}} = j$ where j is the symbol in the wth position of $\mathcal{F}_i$.

Clearly $\psi(x) \in X_{[n]}$ for each $x \in \bar{X}$. We need only show that ψ is onto. Our argument is similar to the one used in Theorem 3.2. Let y be any point in $X_{[n]}$. We will show that there is a point $x \in \bar{X}$ with $\psi(x)_{D_{kM}} = y_{D_{kM}}$. A standard compactness argument then implies that ϕ is onto.

To construct x, let $E = \{(-1, a_2)\} \cup \{(a_1, -1)\}$ where $-1 \leq a_1, a_2 \leq kM - 1$. Choose $\bar{x} \in \bar{X}$ so that $\bar{x}_E$ has the form shown in Figure 5. (As usual, our notation does not indicate that each s can be any of the symbols from $\mathcal{A}$.) We will extend the configuration on $\bar{x}_E$ to obtain the desired $x \in \bar{X}$ with $\psi(x)_{D_{kM}} = y_{D_{kM}}$.

Now consider the corner configuration $\mathcal{C}$ of the type shown in Figure 2 in the lower left corner of Figure 5. There are at least n^{M^2} allowable choices of the type shown in Figure 4 for the configuration appearing in the locations determined by D_M. If $y_{D_M} = \mathcal{F}_i$, the ith configuration from $S(X_{[n]}, D_M)$ in lexicographic ordering, choose a configuration from $\mathcal{C}_i$ for the locations determined by D_M. Continue in this way for all translates $D_M + Mv \subset D_{kM}$, $v \in \mathbb{N}^2$, giving an admissible configuration in $\bar{X}$. Extend this to give the desired $x \in \bar{X}$. □

5. *Entropy and open questions*

We conclude with examples which illustrate two important open questions. First note that, when determining if a system (X, σ_d) can factor onto $(X_{[n]}, \sigma_d)$, it is necessary to have $h(X) \geq \log n$. Calculating entropy for higher-dimensional shifts of finite type is itself a

$$\overline{x}_E = \begin{array}{l} s_c \\ s^{M-1} \\ \vdots \\ s^1 \\ \vdots \\ s_c \\ s^{M-1} \\ \vdots \\ s^1 \\ s_c \;\; s_1\, s_2 \ldots s_{M-1}\; s_c \;\; \ldots \;\; s_1\, s_2 \ldots s_{M-1}\; s_c \end{array}$$

FIGURE 5. The configuration appearing in $\overline{x}$ in locations determined by subset E.

difficult problem, so before we begin our examples we will indicate the methods we used to obtain our entropy estimates.

In **[MP]** the authors provide a method for obtaining entropy estimates for the class of shifts of finite type for which $A_h A_v^{\mathrm{T}} \overset{+}{=} A_v^{\mathrm{T}} A_h$ and $A_h A_v \overset{+}{=} A_v A_h$ where $\overset{+}{=}$ indicates the two matrices have non-zero entries in the same places. This requirement is not overly restrictive; it ensures that admissible rectangular blocks and admissible L-shaped blocks can be extended to arrays in the shift of finite type, thus avoiding some of the undecidability issues mentioned earlier. We will refer to a shift of finite type with these properties as a Markley–Paul shift of finite type.

THEOREM 5.1. **[MP]** *Let (X, σ_2) be a Markley–Paul shift of finite type. Then for any $k \geq 1$,*

$$h(X) \leq \frac{h(X_k)}{k}$$

where (X_k, σ_1) is the one-dimensional shift of finite type consisting of horizontal sequences of height k occurring in (X, σ_2).

If A_h is symmetric then also

$$\frac{h(X_k) - h(X_1)}{k-1} \leq h(X).$$

In the case that (X, σ_2) is corner gluing, there is an alternate lower bound on entropy which does not require symmetry.

THEOREM 5.2. *If (X, σ_2) is a corner gluing (with constant $l > 0$), Markley–Paul shift of finite type, then for all $k \geq 1$*

$$\frac{h(X_k)}{k+l} \leq h(X) \leq \frac{h(X_k)}{k}.$$

Proof. The inequality $h(X) \leq h(X_k)/k$ follows from the previous theorem. We need only verify that $h(X_k)/(k+l) \leq h(X)$.

(X_k, σ_1) is the one-dimensional shift of finite type consisting of horizontal sequences of height k occurring in (X, σ_2). Thus $|S(X_k, D_{m(k+l)})|$ is the number of configurations that can appear on a subset of $\mathbb{Z}^2$ of length $m(k+l)$ and height k. Given two such configurations, $\mathcal{S}_1$ and $\mathcal{S}_2$, from $S(X_k, D_{m(k+l)})$, one can use the corner gluing property to find an allowable configuration on a $m(k+l) \times 2(k+l)$ subset of $\mathbb{Z}^2$. Continuing in this way we can find an allowable configuration on a $m(k+l) \times m(k+l)$ subset of $\mathbb{Z}^2$; this larger configuration has m configurations from $S(X_k, D_{m(k+l)})$ stacked on top of each other, with the gap of width l between each of them filled in an allowable way. Thus we get

$$|S(X, D_{m(k+l)})| \geq |S(X_k, D_{m(k+l)})|^m$$

and

$$\begin{aligned} h(X) &= \lim_{m\to\infty} \frac{1}{(m(k+l))^2} \log |S(X, D_{m(k+l)})| \\ &\geq \lim_{m\to\infty} \frac{1}{(m(k+l))^2} \log |S(X_k, D_{m(k+l)})|^m \\ &= \left(\lim_{m\to\infty} \frac{m}{m(k+l)}\right)\left(\lim_{m\to\infty} \frac{1}{m(k+l)} \log |S(X_k, D_{m(k+l)})|\right) \\ &= \frac{1}{k+l} h(X_k). \end{aligned}$$

□

Example 5.3. Consider the two-dimensional shift of finite type given by the following adjacency matrices:

$$A_h = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad A_v = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This example illustrates that the corner condition is not necessary for factoring onto the full shift. This example does not have corner condition two and yet a subshift of this example is conjugate to a shift of finite type with corner condition two.

Question. Does such a subshift exist (as it does in one dimension) for all shifts of finite type with sufficient entropy?

It is easily verified that this example is a Markley–Paul shift of finite type. This example has a 'safe symbol', the symbol 1, so it is corner gluing with $l = 1$. Using Theorem 5.2, a computer, and $k = 3$, we obtain $0.6968 \leq h(X) \leq 0.8711$. Thus $h(X) > \log 2$.

To check whether this example satisfies the corner condition for $n = 2$ we calculate

$$A_h A_v^{\mathrm{T}} = \begin{pmatrix} 4 & 2 & 2 & 1 \\ 4 & 2 & 2 & 1 \\ 4 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix}.$$

$A_h A_v^{\mathrm{T}}(a, b)$ gives the number of allowable symbols for c in configuration ${}^{a\ c}_{\cdot\ b}$. Although some corner configurations have as many as four corner choices, not all have at least two choices so the corner condition is not satisfied.

Nevertheless, this example does factor onto the full two-dimensional two shift $(X_{[2]}, \sigma)$. Create a map Φ on the 1×2 blocks occurring in X via by

$$\Phi\begin{pmatrix}1\\1\end{pmatrix} = 1_1, \quad \Phi\begin{pmatrix}s\\1\end{pmatrix} = 1_2 \text{ for } s \neq 1 \quad \text{and} \quad \Phi\begin{pmatrix}t\\s\end{pmatrix} = s \text{ for } s \neq 1.$$

This will extend to a factor map ϕ from (X, σ) to the shift of finite type $(\bar{X}, \sigma)$ given by the matrices $\bar{A}_h$ and $\bar{A}_v$ shown below:

$$\bar{A}_h = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}, \quad \bar{A}_v = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The map ϕ is a conjugacy with the 'drop the subscripts' map as its inverse. The shift of finite type $(\bar{X}, \sigma)$ also does not satisfy corner condition two, but the subspace obtained by removing symbol 4 does have corner condition two.

Example 5.4. Consider the two-dimensional shift of finite type given by the following adjacency matrices:

$$A_h = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad A_v = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This example is the finite-to-one factor of a shift of finite type that factors onto the full shift. It is unknown whether the example itself factors onto the full shift.

Question. Is the finite extension necessary or does (X, σ) itself map onto the full shift?

This example is a Markley–Paul shift of finite type. Using Theorem 5.1, a computer and $k = 3$, we obtain $0.7788 \leq h(X) \leq 0.8178$ and $h(X) > \log 2$. This example does not satisfy the corner condition for $n = 2$. We do not know whether a factor map exists onto $(X_{[2]}, \sigma_2)$. However, because it has a safe symbol, it is corner gluing, and so by Theorem 4.2, there exists $M > 1$ such that (X^M, σ_2^M) satisfies the corner condition for $n = 2^{M^2}$. Then, by Theorem 4.3, this example is the M^2-to-one factor of a shift of finite type that does map onto $(X_{[2]}, \sigma_2)$.

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