Speedups and orbit equivalence of finite extensions of ergodic $\mathbb{Z}^d$-actions

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ABSTRACT. We classify n-point extensions of ergodic $\mathbb{Z}^d$-actions up to relative orbit equivalence and establish criteria under which one n-point extension of an ergodic $\mathbb{Z}^d$-action can be sped up to be relatively isomorphic to an n-point extension of another ergodic $\mathbb{Z}^d$-action. Both results are characterized in terms of an algebraic object associated to each n-point extension which is a conjugacy class of subgroups of the symmetric group on n elements.

Contents

1. Introduction

In a 1985 paper of Arnoux, Ornstein, and Weiss [AOW], it is shown that for any two measure-preserving transformations $(X, \mathcal{X}, \mu, T)$ and $(Y, \mathcal{Y}, \nu, S)$ where T is ergodic and S is aperiodic, then one can find a measurable function $p : X \to \mathbb{N}$ such that, by setting $\overline{T}(x) = T^{p(x)}(x)$, $(X, \mathcal{X}, \mu, \overline{T})$ is isomorphic to $(Y, \mathcal{Y}, \nu, S)$. In other words, it is always possible to "speed up" one such transformation to "look like" another. This idea was extended in [BBF] to both group and n-point extensions. In this paper Babichev, Burton, and Fieldsteel showed that for extensions by a locally compact,

second countable group, the function p can be taken to be measurable with respect to a factor. They also consider n-point extensions of the form

$$U : X \times \{1, \ldots, n\} \to X \times \{1, \ldots, n\}$$

where $U^n(x, i) = (T^n x, \sigma(x, n)(i))$ and $\sigma : X \times \mathbb{Z} \to S_n$, the symmetric group on n elements. They use a conjugacy class of subgroups of S_n associated to each such U, originally studied by Mackey [M] and Zimmer [Z1], and use it to characterize which n-point extensions one can relatively speed up to look like another.

Classifications of n-point extensions up to relative equivalence has been performed in other contexts as well. Finite extensions of Bernoulli shifts are classified up to factor isomorphism in [R] and n-point extensions of ergodic automorphisms are classified up to factor orbit equivalence in [G]. The latter work characterizes those which are factor orbit equivalent by defining something called the "G-interchange property".

The works mentioned above concern dynamical systems generated by a single transformation, i.e., actions of $\mathbb{Z}$. It is thus natural to ask what happens when one generalizes to a $\mathbb{Z}^d$-action. In this paper we consider the questions of when one can "speed up" one n-point extension of a $\mathbb{Z}^d$-action to "look like" another, and when two n-point extensions of $\mathbb{Z}^d$-actions are relatively orbit equivalent. As noted in [JM], it is not clear what "speed up" means when there is no "up". As we will define more explicitly below, we will take this to mean that the measurable function $\mathbf{p}$ is now a function from X to $(\mathbb{Z}^d)^d$, and that each coordinate $\mathbf{p}_i$ of $\mathbf{p}$ can be taken so that $\mathbf{p}_i : X \to \mathbf{C}$ where $\mathbf{C}$ is a cone-like region in $\mathbb{Z}^d$. We will call this a $\mathbf{C}$-speedup, and if $\mathbf{p}$ is measurable with respect to a factor, we will call the $\mathbf{C}$-speedup "relative".

A crucial step in understanding which n-point extensions are orbit equivalent and which ones we can speed up to look like others involves associating to each n-point extension $\widetilde{\mathbf{T}}$ an algebraic object we call $gp(\widetilde{\mathbf{T}})$. This is done in Section 3, and it relates the conjugacy class of subgroups of S_n used in [BBF] to the G-interchange property used in [G]. We show the following:

Theorem 1.1. *Let $(X, \mathcal{X}, \mu, \mathbf{T})$ be an ergodic $\mathbb{Z}^{d_1}$-action and let $(Y, \mathcal{Y}, \nu, \mathbf{S})$ be an ergodic $\mathbb{Z}^{d_2}$-action. Suppose these actions have respective n-point extensions $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{S}}$. Then $\widetilde{\mathbf{S}}$ is relatively orbit equivalent to $\widetilde{\mathbf{T}}$ if and only if $gp(\widetilde{\mathbf{S}}) = gp(\widetilde{\mathbf{T}})$.*

Theorem 1.2. *Let $(X, \mathcal{X}, \mu, \mathbf{T})$ and $(Y, \mathcal{Y}, \nu, \mathbf{S})$ be ergodic $\mathbb{Z}^d$-actions with respective n-point extensions $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{S}}$. Given any cone $\mathbf{C}$, there is a relative $\mathbf{C}$-speedup of $\widetilde{\mathbf{T}}$ which is relatively isomorphic to $\widetilde{\mathbf{S}}$ exactly when for each $G_T \in gp(\widetilde{\mathbf{T}})$, there exists $G_S \in gp(\widetilde{\mathbf{S}})$ such that $G_S \subseteq G_T$.*

We begin by providing necessary background definitions and results in Section 2. The definition and properties of $gp(\widetilde{\mathbf{T}})$ are given in Section 3, along with results on orbit equivalence and the proof of Theorem 1.1. The

proof of Theorem 1.2 is given in Section 4, and we conclude in Section 5 with some examples that explore the relationship between $gp(\widetilde{\mathbf{T}})$ and $gp(\widetilde{\mathbf{T}}_{\mathbf{v}})$ where $\widetilde{\mathbf{T}}_{\mathbf{v}}$ is a subaction of $\widetilde{\mathbf{T}}$.

2. Background and definitions

2.1. $\mathbb{Z}^d$-actions. Let X be a Lebesgue probability space with measure μ. Given d commuting, invertible, measurable, measure-preserving transformations $T_1, T_2, \ldots, T_d$ of X, the collection $\{T_j\}$ generates a $\mathbb{Z}^d$-*action* $\mathbf{T}$ on X. In particular, given vector $\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{Z}^d$ we write $\mathbf{T}_{\mathbf{v}}$ for the transformation $T_1^{v_1} \circ T_2^{v_2} \circ \cdots \circ T_d^{v_d} : X \to X$. The $\mathbb{Z}^d$-*action* $\mathbf{T}$ is said to be *ergodic* if the only sets invariant under every $\mathbf{T}_{\mathbf{v}}, \mathbf{v} \in \mathbb{Z}^d$, are of zero or full measure, and is called *totally ergodic* if for every $\mathbf{v} \in \mathbb{Z}^d$, $\mathbf{v} \neq \mathbf{0}$, the transformation $\mathbf{T}_{\mathbf{v}}$ is ergodic.

2.2. Finite extensions. Let $[n] = \{1, 2, 3, \ldots, n\}$; let $2^{[n]}$ denote the power set of $[n]$ and let δ_n be uniform counting measure on $[n]$. Throughout this paper, S_n denotes the symmetric group on n letters.

Definition 2.1. Let $(X, \mathcal{X}, \mu, \mathbf{T})$ be a measure-preserving (m.p.) $\mathbb{Z}^d$-action. A *finite* or *n-point extension* of $(X, \mathcal{X}, \mu, \mathbf{T})$ is another $\mathbb{Z}^d$ m.p. system

$$(X \times [n], \mathcal{X} \times 2^{[n]}, \mu \times \delta_n, \widetilde{\mathbf{T}})$$

defined by setting $\widetilde{\mathbf{T}}_{\mathbf{v}}(x, i) = (\mathbf{T}_{\mathbf{v}}(x), \sigma(x, \mathbf{v})i)$ where $\sigma : X \times \mathbb{Z}^d \to S_n$ is a measurable function satisfying

$$(2.1) \qquad \sigma(x, \mathbf{v} + \mathbf{w}) = \sigma(\mathbf{T}_{\mathbf{v}}(x), \mathbf{w})\sigma(x, \mathbf{v}) \qquad \forall \mathbf{v}, \mathbf{w} \in \mathbb{Z}^d, \text{a.e. } x \in X.$$

The function σ is called the *cocycle* of $\widetilde{\mathbf{T}}$, and Equation (2.1) above is called the *cocycle equation*.

2.3. Group extensions. Let G be a locally compact, second countable group; let λ be Haar measure on G (λ need not be finite).

Definition 2.2. Let $(X, \mathcal{X}, \mu, \mathbf{T})$ be a measure-preserving (m.p.) $\mathbb{Z}^d$-action. A *G-extension* of $(X, \mathcal{X}, \mu, \mathbf{T})$ is another $\mathbb{Z}^d$ m.p. system

$$(X \times G, \mathcal{X} \times \mathcal{G}, \mu \times \lambda, \mathbf{T}^{\sigma})$$

defined by setting

$$\mathbf{T}_{\mathbf{v}}^{\sigma}(x, g) = (\mathbf{T}_{\mathbf{v}}(x), \sigma(x, \mathbf{v})g)$$

for each $\mathbf{v} \in \mathbf{Z}^d$, where $\sigma : X \times \mathbb{Z}^d \to G$ is a measurable function satisfying (2.1). The $\mathbb{Z}^d$-action $\mathbf{T}$ is then referred to as the *base* or *base factor* of $\mathbf{T}^{\sigma}$.

In fact, a locally compact, second-countable group G admits an ergodic G-extension if and only if G is amenable [H], [Z2].

Definition 2.3. Let $\widetilde{\mathbf{T}}$ be an n-point extension of $\mathbf{T}$ with cocycle σ. Then the S_n-extension of $\mathbf{T}$, $\mathbf{T}^\sigma : X \times S_n \to X \times S_n$, defined by

$$\mathbf{T}^\sigma_{\mathbf{v}}(x, g) = (\mathbf{T}_{\mathbf{v}}(x), \sigma(x, \mathbf{v})g)$$

is called the *full extension* or S_n-*extension associated to* $\widetilde{\mathbf{T}}$.

In the setting of either a finite extension or G-extension, we can also define a cocycle on the orbit relation of $\mathbf{T}$, which is again labelled σ: if $z = \mathbf{T}_{\mathbf{v}}(x)$ for some $\mathbf{v} \in \mathbb{Z}^d$, we set $\sigma(x, z) = \sigma(x, \mathbf{v})$.

In this paper we use the symbol σ to refer to most of our cocycles and when necessary, distinguish between the cocycles for different actions with subscripts (i.e., $\sigma_{\mathbf{T}}$ is the cocycle associated to an n-point or S_n-extension of $\mathbf{T}$).

2.4. Iterates and speedups. We define a *filled cone* $\mathbf{C}$ to be any open, connected subset of $\mathbb{R}^d$ whose boundary is contained in d distinct hyperplanes passing through the origin. For example, the interior of the first quadrant is a filled cone in $\mathbb{R}^2$, and the set of points (x, y, z) satisfying $x > 0, y > x$ and $z > y$ is a filled cone in $\mathbb{R}^3$. A *cone* is the intersection of a filled cone with $(\mathbb{Z}^d - \{\mathbf{0}\})$. In particular, notice the zero vector does not belong to any cone.

Given a $\mathbb{Z}^d$-action $(X, \mathcal{X}, \mu, \mathbf{T})$, an *iterate* of $\mathbf{T}$ is an element of the full group of $\mathbf{T}$. In other words, it is a $1-1$, measurable and measure-preserving function $R : X \to X$ given by $R(x) = \mathbf{T}_{\mathbf{k}(x)}(x)$ for some measurable function $\mathbf{k} : X \to \mathbb{Z}^d$. The function $\mathbf{k}$ is called the *iterate function* (of R). If the iterate function of R takes values in a cone $\mathbf{C}$, then we call R a $\mathbf{C}$-*iterate*.

Definition 2.4. Given two $\mathbb{Z}^d$−actions $(Z, \mathcal{Z}, \zeta, \mathbf{U})$ and $(Z, \mathcal{Z}, \zeta, \overline{\mathbf{U}})$, and given a cone $\mathbf{C}$, we say $\overline{\mathbf{U}}$ is a $\mathbf{C}$-*speedup* (or just *speedup*) of $\mathbf{U}$ if there is a measurable map $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_d) : Z \to \mathbf{C}^d$ such that the iterates $\mathbf{U}_{\mathbf{p}_1}, \ldots, \mathbf{U}_{\mathbf{p}_d}$ commute and generate $\overline{\mathbf{U}}$, i.e., $\overline{\mathbf{U}}_{\mathbf{e}_j} = \mathbf{U}_{\mathbf{p}_j}$ for each j. $\mathbf{p}$ is called the *speedup function* of $\overline{\mathbf{U}}$.

As we mentioned in Section 1, the word "speedup" is used in analogy to the 1-dimensional results from [AOW] and [BBF].

If we are considering a finite extension $\widetilde{\mathbf{T}}$ or group extension $\mathbf{T}^\sigma$, then a *relative speedup* of such an action is a speedup whose speedup function is measurable with respect to the base factor.

2.5. Orbit equivalence. Suppose we are given a $\mathbb{Z}^{d_1}$-action $(X, \mathcal{X}, \mu, \mathbf{T})$ and a $\mathbb{Z}^{d_2}$-action $(Y, \mathcal{Y}, \nu, \mathbf{S})$. We say $\mathbf{T}$ and $\mathbf{S}$ are *orbit equivalent* if there is an isomorphism ϕ of the measure spaces $(X, \mathcal{X}, \mu)$ and $(Y, \mathcal{Y}, \nu)$ which preserves orbits, i.e., if $x_2 = \mathbf{T}_{\mathbf{v}}(x_1)$ for some $\mathbf{v} \in \mathbb{Z}^{d_1}$, then $\phi(x_2) = \mathbf{S}_{\mathbf{w}}(\phi(x_1))$ for some $\mathbf{w} \in \mathbb{Z}^{d_2}$ (and similarly for ϕ^{-1}).

Two finite extensions are called *relatively orbit equivalent* if they are orbit equivalent via a map Φ which is measurable with respect to the base factors,

i.e., for all measurable $B \in \mathcal{Y}$, $\Phi^{-1}(B \times [n]) = A \times [n]$ a.s. for some $A \in \mathcal{X}$. Similarly, two group extensions by G are called *relatively orbit equivalent* if they are orbit equivalent via Φ such that for all measurable $B \in \mathcal{Y}$, $\Phi^{-1}(B \times G) = A \times G$ a.s. for some $A \in \mathcal{X}$.

Dye [D1, D2] proved that any ergodic action of $\mathbb{Z}^d$ is orbit equivalent to any ergodic action of $\mathbb{Z}$, and later Fieldsteel [F] proved a relative version of Dye's theorem, showing that any two group extensions by a compact group are relatively orbit equivalent. In 1987 Gerber [G] proved that two n-point extensions of ergodic $\mathbb{Z}$-actions are orbit equivalent if they have the "G-interchange property" for the same group G; in this paper we show how the Gerber result naturally extends to $\mathbb{Z}^d$-actions.

2.6. Isomorphisms. Let $(X, \mathcal{X}, \mu, \mathbf{T})$ and $(Y, \mathcal{Y}, \nu, \mathbf{S})$ be two measure-preserving $\mathbb{Z}^d$-actions with respective G-extensions $\mathbf{T}^\sigma$ and $\mathbf{S}^\sigma$. We say $\mathbf{S}^\sigma$ and $\mathbf{T}^\sigma$ are *G-isomorphic* if there is a measure space isomorphism

$$\Phi : (X \times G, \mathcal{X} \times \mathcal{G}, \mu \times \lambda) \to (Y \times G, \mathcal{Y} \times \mathcal{G}, \nu \times \lambda)$$

that intertwines the dynamics (i.e., $\Phi \circ \mathbf{T}^\sigma_\mathbf{v} = \mathbf{S}^\sigma_\mathbf{v} \circ \Phi$ a.s. for each $\mathbf{v} \in \mathbb{Z}^d$) and is measurable with respect to the base factors. Equivalently, this means the base transformations are isomorphic via some isomorphism ϕ, and that the cocycles are cohomologous after the spaces are identified by ϕ, i.e.,

$$(2.2) \qquad\qquad \Phi(x, g) = (\phi(x), \alpha(x)g)$$

where ϕ is an isomorphism from $(X, \mathcal{X}, \mu, \mathbf{T})$ to $(Y, \mathcal{Y}, \nu, \mathbf{S})$ and $\alpha : X \to G$ is measurable. The α in the previous sentence is called the *transfer function* relating the cocycles. In particular, if $\mathbf{T}^\sigma$ is G-isomorphic to $\mathbf{S}^\sigma$ by the map Φ described above, then the cocycle $\sigma_\mathbf{S}$ must satisfy

$$\sigma_\mathbf{S}(\phi(x), \mathbf{v}) = \alpha(\mathbf{T}_\mathbf{v}x)\sigma_\mathbf{T}(x, \mathbf{v})\alpha(x)^{-1}.$$

Motivated by this fact, if $\mathbf{T}^\sigma$ is a G-extension of a $\mathbb{Z}^d$ action and $\alpha : X \to G$ is any measurable function, we define the *skewing of σ by α* to be the cocycle

$$\sigma^\alpha(x, \mathbf{v}) = \alpha(\mathbf{T}_\mathbf{v}x)\sigma(x, \mathbf{v})\alpha(x)^{-1}$$

and remark that $\mathbf{T}^\sigma$ is $G-$isomorphic to $\mathbf{T}^{(\sigma^\alpha)}$ (the "skewing of $\mathbf{T}^\sigma$ by α") by the map $(x, g) \mapsto (x, \alpha(x)g)$.

We can also talk about relative isomorphisms of finite extensions: given measure-preserving $\mathbb{Z}^d$-actions $(X, \mathcal{X}, \mu, \mathbf{T})$ and $(Y, \mathcal{Y}, \nu, \mathbf{S})$ with respective n-point extensions $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{S}}$, we say $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{S}}$ are *relatively isomorphic* if there is a measure space isomorphism $\Phi : X \times [n] \to Y \times [n]$ which intertwines the dynamics and is measurable with respect to the base factors. The map Φ is called a *relative isomorphism*. Equivalently, this means there is an isomorphism ϕ from $(X, \mathbf{T})$ to $(X, \mathbf{S})$ and a measurable transfer function $\alpha : X \to S_n$ such that

$$(2.3) \qquad\qquad \Phi(x, i) = (\phi(x), \alpha(x)i).$$

In this paper we will need to convert between maps on $X \times S_n$ and maps on $X \times [n]$. Toward that end, we establish the following notation:

Definition 2.5. Given a function $\Phi : X \times [n] \to Y \times [n]$ of the form $\Phi(x,i) = (\phi(x), \alpha(x)i)$ where $\phi : X \to Y$ and $\alpha : X \to S_n$, define Φ^* to be the function

$$\Phi^* : X \times S_n \to Y \times S_n$$

given by $\Phi^*(x,g) = (\phi(x), \alpha(x)g)$.

Given a function $\Phi : X \times S_n \to Y \times S_n$ of the form $\Phi(x,g) = (\phi(x), \alpha(x)g)$ where $\phi : X \to Y$ and $\alpha : X \to S_n$, define Φ_* to be the function

$$\Phi_* : X \times [n] \to Y \times [n]$$

given by $\Phi_*(x,i) = (\phi(x), \alpha(x)i)$.

In either of these settings we call Φ^* or Φ_* the *associated map* of Φ.

In light of the characterizations of G- and relative isomorphisms given in (2.2) and (2.3), we see that an associated map of a relative isomorphism is a relative isomorphism (and record this observation in the following lemma).

Lemma 2.6. *Let* $(X, \mathcal{X}, \mu, \mathbf{T})$ *and* $(Y, \mathcal{Y}, \nu, \mathbf{S})$ *be* $\mathbb{Z}^d$-*actions with respective* n-*point extensions* $\widetilde{\mathbf{T}}$, $\widetilde{\mathbf{S}}$ *whose full extensions are* $\mathbf{T}^\sigma$ *and* $\mathbf{S}^\sigma$.

(1) *If* $\Phi : X \times [n] \to Y \times [n]$ *is a relative isomorphism between* $\widetilde{\mathbf{T}}$ *and* $\widetilde{\mathbf{S}}$, *then* $\Phi^* : X \times S_n \to Y \times S_n$ *is an* S_n-*isomorphism between* $\mathbf{T}^\sigma$ *and* $\mathbf{S}^\sigma$.

(2) *If* $\Phi : X \times S_n \to Y \times S_n$ *is an* S_n-*isomorphism between* $\mathbf{T}^\sigma$ *and* $\mathbf{S}^\sigma$, *then* $\Phi_* : X \times [n] \to Y \times [n]$ *is a relative isomorphism between* $\widetilde{\mathbf{T}}$ *and* $\widetilde{\mathbf{S}}$.

The following theorem, proven in [JM], makes use of a G-isomorphism to relate two G-extensions under certain assumptions:

Theorem 2.7. *Fix a locally compact, second countable group* G *and a neighborhood* $U \subseteq G$ *of the identity element of* G. *Let* $(X, \mathcal{X}, \mu, \mathbf{T})$ *and* $(Y, \mathcal{Y}, \nu, \mathbf{S})$ *be measure-preserving* $\mathbb{Z}^d$-*actions with* $(Y, \mathcal{Y}, \nu, \mathbf{S})$ *aperiodic. Let* $\mathbf{T}^\sigma$ *be an ergodic* G-*extension of* $\mathbf{T}$ *and* $\mathbf{S}^\sigma$ *be a* $G-$*extension of* $\mathbf{S}$. *Let* $\mathbf{C} \subseteq \mathbb{Z}^d$ *be any cone.*

Then there is a relative $\mathbf{C}$-*speedup* $\overline{\overline{\mathbf{T}}}^\sigma$ *of* $\mathbf{T}^\sigma$, *such that* $\overline{\overline{\mathbf{T}}}^\sigma$ *is* G-*isomorphic to* $\mathbf{S}^\sigma$ *via a* G-*isomorphism whose transfer function* $\overline{\alpha}$ *takes values in* U *almost surely.*

In dimension one, a result of [N] states that when the speedup function p is integrable, then the entropies of T and T^p satisfy $h(T^p) = (\int p\, du)\, h(T)$. How this statement generalizes to higher dimensions is an open question.

The proof of Theorem 2.7 in [JM] can be straightforwardly adapted to give the following more general result, which we will use in Section 4:

Theorem 2.8. *Fix a locally compact, second countable group G and a neighborhood $U \subseteq G$ of the identity element of G. Let $(X, \mathcal{X}, \mu, \mathbf{T})$ and $(Y, \mathcal{Y}, \nu, \mathbf{S})$ be measure-preserving actions of $\mathbb{Z}^{d_1}$ and $\mathbb{Z}^{d_2}$, respectively, with $(Y, \mathcal{Y}, \nu, \mathbf{S})$ aperiodic. Let $\mathbf{T}^\sigma$ be an ergodic G-extension of $\mathbf{T}$ and $\mathbf{S}^\sigma$ be a G-extension of $\mathbf{S}$. Let $\mathbf{C} \subseteq \mathbb{Z}^{d_1}$ be any cone.*

Then there are d_2 commuting $\mathbf{C}$-iterates $\overline{T}_1^{\overline{\sigma}}, \ldots, \overline{T}_{d_2}^{\overline{\sigma}}$ of $\mathbf{T}^\sigma$ which generate a $\mathbb{Z}^{d_2}$-action $\overline{\mathbf{T}}^{\overline{\sigma}}$ such that $\overline{\mathbf{T}}^{\overline{\sigma}}$ is G-isomorphic to $\mathbf{S}^\sigma$ via a G-isomorphism whose transfer function $\overline{\alpha}$ takes values in U almost surely.

3. An algebraic invariant associated to orbit equivalence of finite extensions

In this section, we describe an algebraic invariant of a finite extension originally studied in [M] and [Z1] and independently discovered in [R]. First, observe the following, which is part of Theorem 3.25 in [Gl]:

Theorem 3.1. *Let G be a compact group and let $\mathbf{T}^\sigma$ be a G-extension of an ergodic $\mathbb{Z}^d$-action $(X, \mathcal{X}, \mu, \mathbf{T})$. Then there is a subgroup $\overline{G} \subseteq G$ and a G-extension $\mathbf{T}^{\sigma'}$ of $\mathbf{T}$, such that:*

(1) $\mathbf{T}^{\sigma'}$ *is G-isomorphic to $\mathbf{T}^\sigma$, via a G-isomorphism of the form*

$$(x, g) \mapsto (x, \alpha(x)g).$$

(2) $X \times \overline{G}$ *is an ergodic component of $\mathbf{T}^{\sigma'}$.*

Moreover, the set of $\overline{G}$ with this property is a conjugacy class of subgroups of G.

In our setting, we call this conjugacy class the "interchange class" because this machinery relates to what Gerber called the "G-interchange property" in [G]. More precisely:

Definition 3.2. Let $(X, \mathcal{X}, \mu, \mathbf{T})$ be an ergodic $\mathbb{Z}^d$-action and let $\widetilde{\mathbf{T}}$ be an n-point extension of $\mathbf{T}$ whose full S_n-extension is $\mathbf{T}^\sigma$. The *interchange class* of $\widetilde{\mathbf{T}}$, denoted $gp(\widetilde{\mathbf{T}})$, is the conjugacy class of subgroups $\overline{G}$ of S_n such that $\mathbf{T}^\sigma$ is S_n-isomorphic to an S_n-extension of $\mathbf{T}$ with ergodic component $X \times \overline{G}$.

We remark that $gp(\widetilde{\mathbf{T}})$ depends only on the orbit relation of $\mathbf{T}$ and the cocycle σ (see Proposition 3.6 below). As such, in [BBF] this object was denoted $gp(T, \sigma)$. We use slightly different notation because in this paper, we are most interested in the application of this object to n-point extensions.

The next theorem provides an equivalent characterization of the interchange class, generalizing the description in [G] to classify finite extensions of ergodic $\mathbb{Z}$-actions up to relative orbit equivalence.

Theorem 3.3. *Let $(X, \mathcal{X}, \mu, \mathbf{T})$ be an ergodic $\mathbb{Z}^d$-action, and let $\sigma : X \to S_n$ be a cocycle. Let $\mathbf{T}^\sigma$ and $\widetilde{\mathbf{T}}$ be the S_n- and n-point extensions of $\mathbf{T}$*

determined by σ. Let $\overline{G} \subseteq S_n$. Then $\overline{G} \in gp(\widetilde{\mathbf{T}})$ if and only if $\mathbf{T}^\sigma$ is G-isomorphic to another S_n-extension $\mathbf{S}^{\sigma\mathbf{s}}$ of some ergodic action $(Y, \mathcal{Y}, \nu, \mathbf{S})$ satisfying the following conditions:

(1) *For every $\mathbf{v} \in \mathbb{Z}^d$, $\sigma_{\mathbf{S}}(y, \mathbf{v}) \in \overline{G}$ a.s.*
(2) *For any $g \in \overline{G}$ and any sets A and B of equal positive measure in Y, there is an iterate R of $\mathbf{S}$, given by iterate function $\mathbf{k}$, such that $R(A) = B$ and for almost every $y \in A$, $\sigma_{\mathbf{S}}(y, \mathbf{k}) = g$.*

Proof. ($\Rightarrow$): Suppose $\overline{G} \subseteq S_n$ where $X \times \overline{G}$ is an ergodic component of $\mathbf{T}^{\sigma'}$, some S_n-extension of $\mathbf{T}$ which is G-isomorphic to $\mathbf{T}^\sigma$. Note that since $X \times \overline{G}$ is invariant under $\mathbf{T}^{\sigma'}$, for a.e. $x \in X$ and $\mathbf{v} \in \mathbb{Z}^d$, we have that $\sigma'(x, \mathbf{v})$ must be in $\overline{G}$. If we take $g \in \overline{G}$ and $A, B \subseteq X$ with $\mu(A) = \mu(B) > 0$, then by the ergodicity of $\mathbf{T}^{\sigma'}$ on $X \times \overline{G}$, there is an iterate R of $\mathbf{T}^{\sigma'}$ mapping $A \times \{id\}$ to $B \times \{g\}$ (see Lemma 3.3 of [JM]). Thus the equivalent condition is satisfied with $Y = X$, $\mathbf{S} = \mathbf{T}$, $\sigma_{\mathbf{S}} = \sigma'$, and this R.

($\Leftarrow$): Let $\mathbf{S}^{\sigma\mathbf{s}}$ be an S_n-extension of $(Y, \mathcal{Y}, \nu, \mathbf{S})$ which is S_n-isomorphic to $\mathbf{T}^\sigma$ and satisfies (1) and (2) with respect to $\overline{G}$. By (1), $Y \times \overline{G}$ is invariant under $\mathbf{S}^{\sigma\mathbf{s}}$ and by (2), $Y \times \overline{G}$ has no nontrivial invariant subsets. Let Φ_1 be a S_n-isomorphism from $(X \times G, \mathbf{T}^\sigma)$ to $(Y \times G, \mathbf{S}^{\sigma\mathbf{s}})$; this isomorphism (since it is a S_n-isomorphism) has the form

$$\Phi_1(x, g) = (\phi_1(x), \beta_1(x)g)$$

for suitable functions ϕ_1 and β_1. Now define $\Phi_2 : Y \times S_n \to X \times S_n$ by

$$\Phi_2(y, g) = (\phi_1^{-1}(y), g);$$

this is a S_n-isomorphism between $\mathbf{S}^{\sigma\mathbf{s}}$ and some other S_n-extension $\mathbf{T}^{\sigma'}$ of $\mathbf{T}$. Observe that $\Phi_2(Y \times \overline{G}) = X \times \overline{G}$ is an ergodic component of $\mathbf{T}^{\sigma'}$, and the composition $\Phi_2 \circ \Phi_1$ gives a S_n-isomorphism from $\mathbf{T}^\sigma$ to $\mathbf{T}^{\sigma'}$ as desired. $\quad\square$

We remark also that for any $G \in gp(\widetilde{\mathbf{T}})$, the action of S_n on the set of right cosets $G \backslash S_n$ is the *Mackey range* of the cocycle σ (see Section 3.5 of [Gl] for a definition of the Mackey range).

The following two results will be used when studying examples constructed in Section 5.

Proposition 3.4. $\widetilde{\mathbf{T}}$ *is ergodic if and only if there is some $G \in gp(\widetilde{\mathbf{T}})$ which is transitive if and only if every $G \in gp(\widetilde{\mathbf{T}})$ is transitive.*

Proof. Suppose $\widetilde{\mathbf{T}}$, the n-point extension of $\mathbf{T}$, is ergodic. Let $G \in gp(\widetilde{\mathbf{T}})$. By Theorem 3.3 and Lemma 2.6, $\widetilde{\mathbf{T}}$ is relatively isomorphic to some $\widetilde{\mathbf{S}}$ whose full extension $\mathbf{S}^{\sigma\mathbf{s}}$ satisfies (1) and (2) of Theorem 3.3. But then $\widetilde{\mathbf{S}}$ is also ergodic, so given sets $A \times \{i\}$ and $B \times \{j\}$, there is an iterate R with iterate function $\mathbf{k}$ such that $R(A \times \{i\})$ intersects $B \times \{j\}$. That is, for a nontrivial set of $y \in A$, $(\mathbf{S}_{\mathbf{k}(y)}y, \sigma_{\mathbf{S}}(y, \mathbf{k}(y)) i) \in B \times \{j\}$, i.e., $\sigma_{\mathbf{S}}(y, \mathbf{k}(y)) i = j$. But by (1) of Theorem 3.3, $\sigma_{\mathbf{S}}(y, \mathbf{k}(y)) \in G$, so for every pair $i, j \in [n]$, there is an element of G that sends i to j and thus this G is transitive.

It remains to show that if there is $G \in gp(\widetilde{\mathbf{T}})$ that is transitive, then $\widetilde{\mathbf{T}}$ is ergodic. Suppose there exists $G \in gp(\widetilde{\mathbf{T}})$ which is transitive. By Theorem 3.3 and Lemma 2.6, $\widetilde{\mathbf{T}}$ is relatively isomorphic to some $\widetilde{\mathbf{S}}$, which has a full extension $\mathbf{S}^{\sigma \mathbf{s}}$ satisfying (1) and (2) of Theorem 3.3. We first show that $\widetilde{\mathbf{S}}$ is ergodic: suppose not, but rather that it has a nontrivial, nonproper invariant subset. Then we can find $A \times \{i\}$ in this invariant subset and $B \times \{j\}$ outside it, where $\nu(A) = \nu(B) > 0$. Since G is transitive, there exists $g \in G$ which maps i to j. By (2) we can find an iterate that takes A to B and iterate function $\mathbf{k}$ with $\sigma_{\mathbf{S}}(y, \mathbf{k}(y)) = g$ for a.e. $x \in A$. But then this iterate maps $A \times \{i\}$ to $B \times \{j\}$, a contradiction. Thus $\widetilde{\mathbf{S}}$ is ergodic and by isomorphism, so is $\widetilde{\mathbf{T}}$. □

Proposition 3.5. *Let $(X, \mathcal{X}, \mu, \mathbf{T})$ be an ergodic $\mathbb{Z}^d$-action. Let $\widetilde{\mathbf{T}}$ be an n-point extension of $\mathbf{T}$ and fix $\mathbf{v} \in \mathbb{Z}^d$. Then for each $G \in gp(\widetilde{\mathbf{T}})$, there exists $H \in gp(\widetilde{\mathbf{T}}_{\mathbf{v}})$ such that H is a subgroup of G.*

Proof. Let $G \in gp(\widetilde{\mathbf{T}})$. Then by the definition of interchange class, the full extension $\mathbf{T}^\sigma$ is S_n-isomorphic to another S_n-extension $\mathbf{T}^{\sigma'}$ which has $X \times G$ as an ergodic component. This S_n-isomorphism induces an S_n-isomorphism between $\mathbf{T}_{\mathbf{v}}^\sigma$ and $\mathbf{T}_{\mathbf{v}}^{\sigma'}$. Applying Theorem 3.1 to the system $(X \times G, \mathbf{T}_{\mathbf{v}}^{\sigma'})$, we find a subgroup $H \subseteq G$ which satisfies $H \in gp(\widetilde{\mathbf{T}}_{\mathbf{v}})$. □

The next result allows us to convert the results of [G] to actions of $\mathbb{Z}^d$ (in fact, the next three results hold for the actions of any amenable group, not just $\mathbb{Z}^d$):

Proposition 3.6. *Suppose $(X, \mathcal{X}, \mu, \mathbf{T})$ and $(Y, \mathcal{Y}, \nu, \mathbf{S})$ are orbit equivalent actions of $\mathbb{Z}^{d_1}$ and $\mathbb{Z}^{d_2}$, respectively, where the orbit equivalence is given by $\phi : X \to Y$. Then if $\sigma_{\mathbf{T}} : X \to S_n$ is a cocycle for $\mathbf{T}$, the function $\sigma_{\mathbf{S}} : Y \to S_n$ defined by*

$$\sigma_{\mathbf{S}}(y, \mathbf{v}) = \sigma_{\mathbf{T}}(\phi^{-1}(y), \phi^{-1}(\mathbf{S}_{\mathbf{v}}(y)))$$

is a cocycle, and if $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{S}}$ are the n-point extensions determined by $\sigma_{\mathbf{T}}$ and $\sigma_{\mathbf{S}}$, we have $gp(\widetilde{\mathbf{T}}) = gp(\widetilde{\mathbf{S}})$.

Proof. That $\sigma_{\mathbf{S}}$ satisfies the cocycle equation is clear. To show $gp(\widetilde{\mathbf{T}}) = gp(\widetilde{\mathbf{S}})$, suppose $G \in gp(\widetilde{\mathbf{T}})$. By definition, there is a measurable transfer function $\alpha : X \to S_n$ such that $X \times G$ is an ergodic component of $(X \times S_n, \mathbf{T}^{(\sigma^\alpha)})$.

Let $\beta = \alpha \circ \phi^{-1}$ and consider the system $(Y \times S_n, \mathbf{S}^{(\sigma^\beta)})$: note this system is S_n-isomorphic to $(Y \times S_n, \mathbf{S}^\sigma)$. Now note that $\Phi : X \times S_n \to Y \times S_n$ defined by $\Phi(x, g) = (\phi(x), g)$ is a relative orbit equivalence between $\mathbf{T}^{(\sigma^\alpha)}$ and $\mathbf{S}^{(\sigma^\beta)}$. Since orbit equivalences preserve ergodic components, and because $X \times G$ is an ergodic component of $\mathbf{T}^{(\sigma^\alpha)}$, we have that $\Phi(X \times G) = Y \times G$ is an ergodic component of $\mathbf{S}^{(\sigma^\beta)}$. Thus $G \in gp(\widetilde{S})$ and therefore $gp(\widetilde{\mathbf{T}}) \subseteq gp(\widetilde{\mathbf{S}})$.

A symmetric argument shows the reverse inclusion and establishes the theorem. $\qquad\square$

Proposition 3.7. *Let* $(X, \mathcal{X}, \mu, \mathbf{T})$ *be any ergodic* $\mathbb{Z}^d$*-action. Given any subgroup* G *of* S_n, *there is a cocycle* $\sigma : X \to S_n$ *for* $\mathbf{T}$ *so that the n-point extension* $\widetilde{\mathbf{T}}$ *determined by* σ *satisfies* $G \in gp(\widetilde{\mathbf{T}})$.

Proof. Let $(G^{\mathbb{Z}}, S)$ be the full shift with alphabet G. Define a cocycle $\sigma_S : G^{\mathbb{Z}} \times \mathbb{Z} \to G$ by

$$\sigma_S(x, n) = x_{n-1} \cdots x_1 x_0$$

where concatenation indicates the group operation. Letting $\widetilde{S}$ be the n-point extension of S defined by this cocycle, we have $G \in gp(\widetilde{S})$ by the proposition on page 34 of [G]. By Dye's theorem, S is orbit equivalent (via some map $\phi : G^{\mathbb{Z}} \to X$) to $\mathbf{T}$. Now define $\sigma_{\mathbf{T}} : X \times \mathbb{Z}^d \to S_n$ setting

$$\sigma_{\mathbf{T}}(x, \mathbf{v}) = \sigma_S(\phi^{-1}(x), \phi^{-1}(\mathbf{T_v}(x))).$$

By Proposition 3.6, $\sigma_{\mathbf{T}}$ is a cocycle with $gp(\widetilde{\mathbf{T}}) = gp(\widetilde{S}) \ni G$. $\qquad\square$

The next result is the first theorem mentioned in the introduction (Theorem 1.1).

Theorem 3.8. *Let* $(X, \mathcal{X}, \mu, \mathbf{T})$ *be an ergodic* $\mathbb{Z}^{d_1}$*-action and let* $(Y, \mathcal{Y}, \nu, \mathbf{S})$ *be an ergodic* $\mathbb{Z}^{d_2}$*-action. Suppose these actions have respective n-point extensions* $\widetilde{\mathbf{T}}$ *and* $\widetilde{\mathbf{S}}$. *Then* $\widetilde{\mathbf{S}}$ *is relatively orbit equivalent to* $\widetilde{\mathbf{T}}$ *if and only if* $gp(\widetilde{\mathbf{S}}) = gp(\widetilde{\mathbf{T}})$.

Proof. $\mathbf{T}$ and $\mathbf{S}$ are orbit equivalent to $\mathbb{Z}$-actions T' and S', respectively, via maps ϕ_T and ϕ_S. Let

$$\sigma'_T(x', n) = \sigma_{\mathbf{T}}(\phi_T^{-1}(x'), \phi_T^{-1}(T'^n(x')))$$

and let $\widetilde{T'}$ be the n-point extension of T' given by σ' (and define $\widetilde{S'}$ similarly). $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{S}}$ are relatively orbit equivalent if and only if $\widetilde{T'}$ and $\widetilde{S'}$ are relatively orbit equivalent if and only if $gp(\widetilde{T'}) = gp(\widetilde{S'})$ (by [G]) if and only if $gp(\widetilde{\mathbf{T}}) = gp(\widetilde{\mathbf{S}})$ (by Proposition 3.6). $\qquad\square$

4. Speedups and relative isomorphisms

In this section we want to consider ergodic n-point extensions of two $\mathbb{Z}^d$-actions, and ask when one can be "sped up" to "look like" the other. This will generalize Theorem 2 of [BBF].

Recall that given an ergodic $\mathbb{Z}$-action $f : X \to X$ and a subset $A \subseteq X$, it is well known how to induce an action $f_A : A \to A$. We begin this section with the following preliminary lemma, which modifies the above notion to our situation.

Lemma 4.1. *Let $(X, \mathcal{X}, \mu, \mathbf{T})$ be an ergodic $\mathbb{Z}^d$-action and let G be a subgroup of S_n. Let $\mathbf{T}^\sigma$ be an ergodic G-extension of $(X, \mathcal{X}, \mu, \mathbf{T})$. Given any cone $\mathbf{C}$ and any subgroup $H \subseteq G$, there is a relative $\mathbf{C}$-iterate of $\mathbf{T}^\sigma$ which has $X \times H$ as an ergodic component.*

Proof. This result is obtained by speeding up the original G-extension $\mathbf{T}^\sigma$ and then picking a particular iterate of this speedup. First, let

$$(Y \times G, \mathcal{Y} \times \mathcal{G}, \nu \times \lambda, \mathbf{S}^\sigma)$$

be any totally ergodic $\mathbb{Z}^d$ G-extension. By Theorem 2.7, there is a relative $\mathbf{C}$-speedup $\overline{\mathbf{T}^\sigma}$ of $\mathbf{T}^\sigma$ which is G-isomorphic to $\mathbf{S}^\sigma$.

Second, let $\mathbf{v} = \mathbf{e}_1$ (the first element of the standard basis of $\mathbb{R}^d$), and consider the ergodic transformation $\left(\overline{\mathbf{T}^\sigma}\right)_{\mathbf{v}}$. Construct the first return time map of $\left(\overline{\mathbf{T}^\sigma}\right)_{\mathbf{v}}$ to $X \times H$, i.e., let $j : X \times H \to \mathbb{N}$ be defined by

$$j(x, h) = \min\left\{i \in \mathbb{N} : \left(\overline{\mathbf{T}^\sigma}\right)_{i\mathbf{v}}(x, h) \in X \times H\right\}.$$

In fact, j depends only on x (and we therefore subsequently write $j(x)$ instead of $j(x, h)$), because for any $(x, h) \in X \times H$ and any $i \in \mathbb{N}$, we have

$$\left(\overline{\mathbf{T}^\sigma}\right)_{i\mathbf{v}}(x, h) \in X \times H \Leftrightarrow \overline{\sigma}(x, i\mathbf{v})h \in H \Leftrightarrow \overline{\sigma}(x, i\mathbf{v}) \in H$$

(since H is a subgroup) and this last condition depends on x but not on h. To complete this second step, define R to the $\mathbf{C}$-iterate given by

$$R(x, h) = \left(\overline{\mathbf{T}^\sigma}\right)_{j(x)\mathbf{v}}(x, h).$$

Composing the speedup functions from these two steps, the transformation R is the relative $\mathbf{C}$-speedup of $\mathbf{T}^\sigma$, as desired. $\qquad\square$

The following result is the main result of our paper; it includes Theorem 1.2 as mentioned in the introduction in the case where $d_1 = d_2$.

Theorem 4.2. *Let $(X, \mathcal{X}, \mu, \mathbf{T})$ and $(Y, \mathcal{Y}, \nu, \mathbf{S})$ be ergodic actions of $\mathbb{Z}^{d_1}$ and $\mathbb{Z}^{d_2}$, respectively, with respective n-point extensions $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{S}}$. Then, the following are equivalent:*

(1) *Given any cone $\mathbf{C}$, there are d_2 commuting $\mathbf{C}$-iterates of $\widetilde{\mathbf{T}}$ which generate a $\mathbb{Z}^{d_2}$-action relatively isomorphic to $\widetilde{\mathbf{S}}$.*

(2) *For some cone $\mathbf{C}$, there are d_2 commuting $\mathbf{C}$-iterates of $\widetilde{\mathbf{T}}$ which generate a $\mathbb{Z}^{d_2}$-action relatively isomorphic to $\widetilde{\mathbf{S}}$.*

(3) *Given any $G_T \in gp(\widetilde{\mathbf{T}})$, there exists $G_S \in gp(\widetilde{\mathbf{S}})$ such that $G_S \subseteq G_T$.*

(4) *For some $G_T \in gp(\widetilde{\mathbf{T}})$, there exists $G_S \in gp(\widetilde{\mathbf{S}})$ such that $G_S \subseteq G_T$.*

Proof. (1) $\Rightarrow$ (2) is obvious; we will first show (2) $\Rightarrow$ (3). We know by assumption that there are d_2 relative $\mathbf{C}$-iterates of $\widetilde{\mathbf{T}}$ which generate a $\mathbb{Z}^{d_2}$-action relatively isomorphic to $\widetilde{\mathbf{S}}$; let $\mathbf{p}_1, \ldots, \mathbf{p}_{d_2}$ be the associated iterate functions of these iterates (see Section 2.4). Note that by Lemma 2.6, the

$\mathbb{Z}^d$-action $\overline{\mathbf{T}}^{\overline{\sigma}}$ generated by the $\mathbf{C}$-iterates of $\mathbf{T}^\sigma$ given by the same iterate functions $\mathbf{p}_1, \ldots, \mathbf{p}_{d_2}$ is S_n-isomorphic to $\mathbf{S}^\sigma$.

Now let $G_T \in gp(\widetilde{\mathbf{T}})$. By Definition 3.2, there is an S_n-extension $\mathbf{T}^{\sigma'}$, S_n-isomorphic to $\mathbf{T}^\sigma$, which has $X \times G_T$ as an ergodic component. Again use the same iterate functions $\mathbf{p}_1, \ldots, \mathbf{p}_{d_2}$ to yield d_2 commuting $\mathbf{C}$-iterates of $\mathbf{T}^{\sigma'}$ which generate a $\mathbb{Z}^{d_2}$ action of $\mathbf{T}^{\sigma'}$, denoted $\overline{\mathbf{T}}^{\overline{\sigma'}}$, which is S_n-isomorphic to $\overline{\mathbf{T}}^{\overline{\sigma}}$.

Composing these two isomorphisms gives a S_n-isomorphism between $\overline{\mathbf{T}}^{\overline{\sigma'}}$ and $\mathbf{S}^\sigma$.

Next, note that $X \times G_T$ is an invariant set for $\overline{\mathbf{T}}^{\overline{\sigma'}}$ and therefore by applying Theorem 3.1, we obtain a subgroup of G_T, which we denote by G_S, such that $\overline{\mathbf{T}}^{\overline{\sigma'}}$ is S_n-isomorphic to another S_n-extension which has ergodic component $X \times G_S$. Composing these isomorphisms, we get that $\mathbf{S}^\sigma$ is S_n-isomorphic to an S_n-extension satisfying (1) and (2) of Theorem 3.3 and thus $G_S \in gp(\widetilde{\mathbf{S}})$, as desired.

Note that $(3) \Rightarrow (4)$ is obvious, so it only remains to show $(4) \Rightarrow (1)$. For this, let $\mathbf{C}$ be a cone. By assumption, there is some $G_T \in gp(\widetilde{\mathbf{T}})$ for which there exists $G_S \in gp(\widetilde{\mathbf{S}})$ with $G_S \subseteq G_T$. By the definition of interchange class, $\mathbf{T}^\sigma$ is S_n-isomorphic to $\mathbf{T}^{\sigma'}$ which has $X \times G_T$ as an ergodic component and $\mathbf{S}^\sigma$ is S_n-isomorphic to $\mathbf{S}^{\sigma'}$ which has $X \times G_S$ as an ergodic component.

Consider $\mathbf{T}^{\sigma'}$ restricted to this ergodic component $X \times G_T$. By Lemma 4.1 we can find a $\mathbf{C}$-iterate R of $\mathbf{T}^{\sigma'}$ (R is an action of $\mathbb{Z}$), which has $X \times G_S$ as an ergodic component. We can now use Theorem 2.8, applied to the restriction of R to $X \times G_S$, to yield a relative $\mathbf{C}$-speedup of R (this speedup is a $\mathbb{Z}^{d_2}$-action) which is G_S-isomorphic to $\mathbf{S}^{\sigma'}$. Composing the two speedup functions, we have a $\mathbf{C}$-speedup of $\mathbf{T}^{\sigma'}$ which is G_S-isomorphic to $\mathbf{S}^{\sigma'}$. Using the same speedup function yields a $\mathbf{C}$-speedup of $\widetilde{\mathbf{T}}$ which by Lemma 2.6 is relatively isomorphic to $\widetilde{\mathbf{S}}$, as desired. $\square$

5. Relationship between the interchange classes of a $\mathbb{Z}^d$-action and its generators

In this section we examine the (lack of) relationship between the interchange class of an n-point extension of a $\mathbb{Z}^d$-action and the interchange classes of its generators and directions. First, the following result demonstrates that the interchange class of a $\mathbb{Z}^d$-action cannot be discerned by looking at the interchange classes of its generators. Recall that $gp(\widetilde{\mathbf{T}}_\mathbf{v})$ is only defined when $\mathbf{T}_\mathbf{v}$ is ergodic; we therefore consider only totally ergodic actions in the next two results.

Proposition 5.1. *For any totally ergodic $\mathbb{Z}^2$-action $\mathbf{T}$, there exist two 4-point extensions $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{T}}'$ such that $gp(\widetilde{\mathbf{T}}_{(1,0)}) = gp(\widetilde{\mathbf{T}}'_{(1,0)})$ and $gp(\widetilde{\mathbf{T}}_{(0,1)}) = gp(\widetilde{\mathbf{T}}'_{(0,1)})$ but $gp(\widetilde{\mathbf{T}}) \neq gp(\widetilde{\mathbf{T}}')$.*

Proof. Define cocycles $\sigma, \sigma' : X \times \mathbb{Z}^2 \to S_4$ by setting

$$\sigma(x, (v_1, v_2)) = (12)^{v_1 + v_2};$$
$$\sigma'(x, (v_1, v_2)) = (12)^{v_1}(34)^{v_2}$$

for all $x \in X$ (i.e., σ and σ' depend only on $\mathbf{v} = (v_1, v_2)$ and not on x). Let $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{T}}'$ be the respective 4-point extensions of $\mathbf{T}$. Note that

$$\langle (12) \rangle \in gp(\widetilde{\mathbf{T}}_{(1,0)}) \quad \text{and} \quad \langle (12) \rangle \in gp(\widetilde{\mathbf{T}}'_{(1,0)}).$$

By Theorem 3.1 this means $gp(\widetilde{\mathbf{T}}_{(1,0)})$ and $gp(\widetilde{\mathbf{T}}'_{(1,0)})$ are the same conjugacy class of subgroups of S_n, i.e., $gp(\widetilde{\mathbf{T}}_{(1,0)}) = gp(\widetilde{\mathbf{T}}'_{(1,0)})$.

Next, note that

$$\langle (12) \rangle \in gp(\widetilde{\mathbf{T}}_{(0,1)}) \quad \text{and} \quad \langle (34) \rangle \in gp(\widetilde{\mathbf{T}}'_{(0,1)});$$

since $\langle (12) \rangle$ and $\langle (34) \rangle$ are conjugate subgroups, Theorem 3.1 again says that $gp(\widetilde{\mathbf{T}}_{(0,1)}) = gp(\widetilde{\mathbf{T}}'_{(0,1)})$.

However,

$$\langle (12) \rangle \in gp(\widetilde{\mathbf{T}}) \quad \text{while} \quad \langle (12), (34) \rangle \in gp(\widetilde{\mathbf{T}}').$$

As $\langle (12) \rangle$ is a subgroup of S_4 of order two and $\langle (12), (34) \rangle$ is a subgroup of S_4 of order four, $gp(\widetilde{\mathbf{T}}) \neq gp(\widetilde{\mathbf{T}}')$. $\square$

Applying Theorems 3.8 and 4.2 to the two extensions described in the previous proposition, we see:

Corollary 5.2. *For any totally ergodic $\mathbb{Z}^2$-action $\mathbf{T}$, there exist two 4-point extensions $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{T}}'$ such that:*

(1) *Each generator of $\widetilde{\mathbf{T}}$ is relatively orbit equivalent to the corresponding generator of $\widetilde{\mathbf{T}}'$, and each generator can be relatively sped up to obtain a relatively isomorphic copy of the respective generator of $\widetilde{\mathbf{T}}'$, but*

(2) *$\widetilde{\mathbf{T}}$ is not relatively orbit equivalent to $\widetilde{\mathbf{T}}'$, and for any cone $\mathbf{C} \subseteq \mathbb{Z}^2$, there is no relative $\mathbf{C}$-speedup of $\widetilde{\mathbf{T}}$ which is relatively isomorphic to $\widetilde{\mathbf{T}}'$.*

We now move to an example illustrating the reverse situation, i.e., when the interchange classes of the $\mathbb{Z}^2$-actions coincide but the properties of the directions of those actions are quite different. We first build an example that, similar to $\widetilde{\mathbf{T}}'$ above, shows how different the interchange class for the action can be from the interchange classes of its generators.

Lemma 5.3. *There exists an ergodic $\mathbb{Z}^2$-action $(X, \mathcal{X}, \mu, \mathbf{T})$ which has an ergodic 2-point extension $\widetilde{\mathbf{T}}$ such that $gp(\widetilde{\mathbf{T}}_{(1,0)}) = gp(\widetilde{\mathbf{T}}_{(0,1)}) = \{id\}$, but $gp(\widetilde{\mathbf{T}}) = S_2$.*

Proof. Let $X = \{0,1\}^{\mathbb{Z}^2} \times [2]$. Then X is a 2-point extension of the full shift on two symbols (i.e., the base space in this example is itself a 2-point extension). Let $\boldsymbol{\Sigma}$ be the usual two-dimensional shift map, and define $\mathbf{T}$ to be the $\mathbb{Z}^2$-action

$$\mathbf{T}_{(v_1, v_2)}(x, i) = \left(\boldsymbol{\Sigma}_{(v_1, v_2)} x, (12)^{v_1 + v_2} i \right).$$

In other words, this action is defined via a cocycle that only depends on the vector (v_1, v_2) and not on x. The cocycle satisfies the cocycle condition and thus the two-dimensional action $\mathbf{T}$ is well-defined. Note that although this action $\mathbf{T}$ is ergodic, and its generators $\mathbf{T}_{(1,0)}$ and $\mathbf{T}_{(0,1)}$ are both ergodic, there are other subactions of $\mathbf{T}$ which are not.

Next define a 2-point extension of $(X, \mathbf{T})$ via the cocycle

$$\sigma \left((x, i),\, (v_1, v_2) \right) = \sigma(v_1, v_2) = (12)^{v_1}.$$

The cocycle σ also satisfies the cocycle condition and thus yields the action $\widetilde{\mathbf{T}}$ on the space $\widetilde{X} = X \times [2]$. We note that

$$\widetilde{\mathbf{T}}_{(1,0)} \left((x, i), j \right) = \left((\boldsymbol{\Sigma}_{(1,0)} x, (12) i),\, (12) j \right)$$

and

$$\widetilde{\mathbf{T}}_{(0,1)} \left((x, i), j \right) = \left((\boldsymbol{\Sigma}_{(0,1)} x, (12) i),\, j \right).$$

The space $\widetilde{X}$ can be divided into four disjoint sets D_{11}, D_{12}, D_{21} and D_{22} where

$$D_{ij} = \left(\{0,1\}^{\mathbb{Z}^2} \times \{i\} \right) \times \{j\}.$$

Observe that the set $D_{11} \cup D_{22}$ is an invariant set for $\widetilde{\mathbf{T}}_{(1,0)}$ while $D_{11} \cup D_{21}$ is an invariant set for $\widetilde{\mathbf{T}}_{(0,1)}$, showing that these two actions are not ergodic. By Proposition 3.4, their interchange classes must contain only non-transitive subgroups of S_2. Therefore $gp(\widetilde{\mathbf{T}}_{(1,0)})$ and $gp(\widetilde{\mathbf{T}}_{(0,1)})$ each consists only of the trivial subgroup $\{id\}$.

However, we can see that $\widetilde{\mathbf{T}}$ is ergodic as follows: let $\widetilde{A}$ and $\widetilde{B}$ be nontrivial subsets of $X \times [2]$. We can then find nontrivial subsets $A, B \subseteq \{0,1\}^{\mathbb{Z}^2}$ and i_1, i_2, j_1, j_2 such that $(A \times i_1) \times \{j_1\} \subseteq A$ and $(B \times i_2) \times \{j_2\} \subseteq B$. If $j_1 = j_2$, we use the structure of $\{0,1\}^{\mathbb{Z}^2}$ to find $(0, k)$, where k is even if $i_1 = i_2$ and odd otherwise, such that $\boldsymbol{\Sigma}_{(0,k)} A$ intersects B nontrivially. This is turn tells us that $\widetilde{\mathbf{T}}_{(0,k)} \widetilde{A}$ intersects $\widetilde{B}$ nontrivially. If $j_1 \neq j_2$, we again use the structure of $\{0,1\}^{\mathbb{Z}^2}$ to find $(1, k)$, where $k + 1$ is even if $i_1 = i_2$ and odd otherwise, so that $\boldsymbol{\Sigma}_{(1,k)} A$ intersects B nontrivially. Again this tells us that $\widetilde{\mathbf{T}}_{(1,k)} \widetilde{A}$ intersects $\widetilde{B}$ nontrivially. Thus $\widetilde{\mathbf{T}}$ is ergodic and by Proposition 3.4, $gp(\widetilde{\mathbf{T}})$ is a conjugacy class of transitive subgroups of S_2, so $gp(\widetilde{\mathbf{T}}) = \{S_2\}$. $\qquad\square$

We now use this example, along with one constructed in the proposition below, to show that it is possible for the relationship between two

2-dimensional actions to be quite different than the relationship between their respective generators.

Proposition 5.4. *There exist two $\mathbb{Z}^2$-actions $(X, \mathcal{X}, \mu, \mathbf{T})$ and $(Y, \mathcal{Y}, \nu, \mathbf{S})$ with respective 2-point extensions $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{S}}$ such that:*

(1) $\widetilde{\mathbf{T}}$ *and* $\widetilde{\mathbf{S}}$ *are relatively orbit equivalent and there is a relative speedup of* $\widetilde{\mathbf{T}}$ *which is relatively isomorphic to* $\widetilde{\mathbf{S}}$; *but*

(2) *given any* $\mathbf{v} \in \mathbb{Z}^2$ *and any* $\mathbf{w} \neq (0, 0)$ *in* $\mathbb{Z}^2$, $\widetilde{\mathbf{T}}_{\mathbf{v}}$ *is not relatively orbit equivalent to* $\widetilde{\mathbf{S}}_{\mathbf{w}}$, *and, for any cone* $\mathbf{C}$, *there is no relative* $\mathbf{C}$-*speedup of* $\widetilde{\mathbf{T}}_{\mathbf{v}}$ *which is relatively isomorphic to* $\widetilde{\mathbf{S}}_{\mathbf{w}}$.

Proof. $\widetilde{\mathbf{S}}$ will be defined as a 2-point extension of a $\mathbb{Z}^2$ subshift of finite type: first, let $\Omega = S_2 \times S_2 = \{id, (12)\}^2$ and define $\pi_1, \pi_2 : \Omega \to S_2$ to be projection onto the first and second coordinate, respectively. Next, let $Y \subseteq \Omega^{\mathbb{Z}^2}$ be the set of 2-dimensional infinite arrays $\{y_{\mathbf{v}} : \mathbf{v} \in \mathbb{Z}^2\}$ of symbols in Ω which satisfy, for every $\mathbf{v} = (v_1, v_2) \in \mathbb{Z}^d$,

(5.1) $$\pi_2(y_{\mathbf{v}+(1,0)})\pi_1(y_{\mathbf{v}}) = \pi_1(y_{\mathbf{v}+(0,1)})\pi_2(y_{\mathbf{v}}).$$

This set Y is invariant under the 2-dimensional shift $\mathbf{S}$, and in fact $\mathbf{S}$ preserves a measure ν on Y which can be informally described as follows: to specify a point $y = \{y_{(i,j)}\}_{(i,j) \in \mathbb{Z}^2} \in Y$, start by taking an i.i.d. sequence of S_2-valued random variables, each uniform on S_2. Think of this sequence as giving the values of $y_{(i,0)}$ where i ranges over $\mathbb{Z}$. Now, independent of this sequence, for each $i \in \mathbb{Z}$ take another sequence of i.i.d. random variables, where each random variable is again uniform on S_2, and think of this sequence as giving the values of $y_{(i,j)}$ where j ranges over $\mathbb{Z}$. Note that the sequences associated to different i's are chosen independently of one another. The coordinates of y so chosen determine the remaining coordinates using (5.1). To define ν, let the measure of any cylinder be the probability that a point y so chosen lies in that cylinder.

Note that under ν, knowing the coordinates at a single index (or even a finite set of indices) of some $y \in Y$ does not affect the coordinates of y that are some distance away, so cylinder sets in Y that are "sufficiently far apart" are independent. It follows that $(Y, \mathcal{Y}, \nu, \mathbf{S})$ is totally ergodic.

Now define a cocycle $\sigma : Y \times \mathbb{Z}^d \to S_2$ by setting

$$\sigma(y, (v_1, v_2)) = \pi_2(y_{(v_1, v_2 - 1)})\pi_2(y_{(v_1, v_2 - 2)}) \cdots \pi_2(y_{(v_1, 0)})\pi_1(y_{(v_1 - 1, 0)}) \cdots$$
$$\cdots \pi_1(y_{(2,0)})\pi_1(y_{(1,0)})\pi_1(y_{(0,0)}).$$

By the definition of Y, this σ satisfies the cocycle condition and therefore determines a 2-point extension $\widetilde{\mathbf{S}}$ of $\mathbf{S}$. Fix any $\mathbf{w} = (w_1, w_2) \in \mathbb{Z}^d$ with $\mathbf{w} \neq (0, 0)$. One can show, using the independence property of ν and the definition of σ, that $\widetilde{\mathbf{S}}_{\mathbf{w}}$ is ergodic. Thus by Proposition 3.4 we have $gp(\widetilde{\mathbf{S}}_{\mathbf{w}}) = \{S_2\}$. It follows that $\widetilde{\mathbf{S}}$ is also ergodic and $gp(\widetilde{\mathbf{S}}) = \{S_2\}$ as well.

Let $\widetilde{\mathbf{T}}$ be the system from Lemma 5.3. In that result we saw that $gp(\widetilde{\mathbf{T}}) = \{S_2\}$, so by Theorems 3.8 and 4.2 we obtain statement (1) of the proposition.

Now let $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^2$ with $\mathbf{w} \neq (0,0)$. We have already said $gp(\widetilde{\mathbf{S}}_{\mathbf{w}}) = \{S_2\}$. To verify statement (2) of this proposition, we consider two cases: when $v_1 + v_2$ is even and $v_1 + v_2$ is odd. First, assume $v_1 + v_2$ is even. Then

$$\mathbf{T}_{(v_1,v_2)}(x,i) = \left(\boldsymbol{\Sigma}_{(v_1,v_2)}x, (12)^{v_1+v_2}i\right) = \left(\boldsymbol{\Sigma}_{(v_1,v_2)}x, i\right)$$

and we see that the action $\mathbf{T}_{(v_1,v_2)}$ has, for instance, $\{0,1\}^{\mathbb{Z}^2} \times \{1\}$ as an invariant set and thus is not ergodic. Since $\mathbf{S}_{\mathbf{w}}$ is ergodic, it is clear that the two base actions $\mathbf{T}_{\mathbf{v}}$ and $\mathbf{S}_{\mathbf{w}}$ are not orbit equivalent and that no speedup of $\mathbf{T}_{\mathbf{v}}$ is isomorphic to $\mathbf{S}_{\mathbf{w}}$. It then follows that $\widetilde{\mathbf{T}}_{\mathbf{v}}$ and $\widetilde{\mathbf{S}}_{\mathbf{w}}$ cannot be relatively orbit equivalent, nor is there a speedup of $\widetilde{\mathbf{T}}_{\mathbf{v}}$ relatively isomorphic to $\widetilde{\mathbf{S}}_{\mathbf{w}}$, yielding statement (2) of the proposition for this case.

Next we assume $v_1 + v_2$ is odd. Then

$$\mathbf{T}_{(v_1,v_2)}(x,i) = \left(\boldsymbol{\Sigma}_{(v_1,v_2)}x, (12)^{v_1+v_2}i\right) = \left(\boldsymbol{\Sigma}_{(v_1,v_2)}x, (12)i\right)$$

This action is ergodic and we proceed to compute $gp(\widetilde{\mathbf{T}}_{\mathbf{v}})$. We again look at two cases: when v_1 is even and when v_1 is odd. In the first case we have

$$\widetilde{\mathbf{T}}_{(v_1,v_2)}((x,i),j) = \left(\left(\boldsymbol{\Sigma}_{(v_1,v_2)}x, (12)^{v_1+v_2}i\right), j\right) = \left(\left(\boldsymbol{\Sigma}_{(v_1,v_2)}x, (12)i\right), j\right)$$

and, similar to how we argued for $\widetilde{\mathbf{T}}_{(0,1)}$ in Lemma 5.3, we see that $D_{11} \cup D_{21}$ is an invariant set for $\widetilde{\mathbf{T}}_{\mathbf{v}}$. Thus $\widetilde{\mathbf{T}}_{\mathbf{v}}$ is not ergodic and $gp(\widetilde{\mathbf{T}}_{\mathbf{v}}) = \{id\}$. On the other hand, if v_1 is odd we have

$$\widetilde{\mathbf{T}}_{(v_1,v_2)}((x,i),j) = \left(\left(\boldsymbol{\Sigma}_{(v_1,v_2)}x, (12)^{v_1+v_2}i\right), (12)j\right)$$
$$= \left(\left(\boldsymbol{\Sigma}_{(v_1,v_2)}x, (12)i\right), (12)j\right)$$

and, again similar to how we argued for $\widetilde{\mathbf{T}}_{(1,0)}$ in Lemma 5.3, we see that $D_{11} \cup D_{22}$ is an invariant set for $\widetilde{\mathbf{T}}_{\mathbf{v}}$. Again we have $\widetilde{\mathbf{T}}_{\mathbf{v}}$ is not ergodic and therefore $gp(\widetilde{\mathbf{T}}_{\mathbf{v}}) = \{id\}$.

In all cases we have $gp(\widetilde{\mathbf{T}}_{\mathbf{v}}) = \{id\}$ while $gp(\widetilde{\mathbf{S}}_{\mathbf{w}}) = \{S_2\}$. Theorem 3.8 then tells us that $\widetilde{\mathbf{T}}_{\mathbf{v}}$ and $\widetilde{\mathbf{S}}_{\mathbf{w}}$ are not relatively orbit equivalent. We use Theorem 4.2 to show that it is not possible to find a relative speedup of $\widetilde{\mathbf{T}}_{\mathbf{v}}$ which is relatively isomorphic to $\widetilde{\mathbf{S}}_{\mathbf{w}}$, yielding statement (2) of the proposition, as wanted. $\square$

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