SPEEDUPS OF ERGODIC GROUP EXTENSIONS OF $\mathbb{Z}^d$-ACTIONS

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ABSTRACT. We define what it means to "speed up" a $\mathbb{Z}^d-$measure-preserving dynamical system, and prove that given any ergodic extension $\mathbf{T}^\sigma$ of a $\mathbb{Z}^d-$ measure-preserving action by a locally compact, second countable group G, and given any second $G-$extension $\mathbf{S}^\sigma$ of an aperiodic $\mathbb{Z}^d-$ measure-preserving action, there is a relative speedup of $\mathbf{T}^\sigma$ which is relatively isomorphic to $\mathbf{S}^\sigma$. Furthermore, we show that given any neighborhood of the identity element of G, the aforementioned speedup can be constructed so that the transfer function associated to the isomorphism between the speedup and $\mathbf{S}^\sigma$ almost surely takes values only in that neighborhood.

1. Introduction

In a 1985 paper of Arnoux, Ornstein, and Weiss [AOW], it is shown that for any ergodic measure-preserving transformation $(X, \mathcal{X}, \mu, T)$ and any aperiodic (not necessarily ergodic) measure-preserving transformation $(Y, \mathcal{Y}, \nu, S)$, one can find a measurable function $p : X \to \mathbb{N}$ such that, by setting $\overline{T}(x) = T^{p(x)}(x)$, $(X, \mathcal{X}, \mu, \overline{T})$ is isomorphic to $(Y, \mathcal{Y}, \nu, S)$. In other words, it is always possible to "speed up" one such transformation to "look like" another. If restrictions are placed on the type of function allowed for p, then the result is also restricted. For instance, Neveu [N] gives a generalized version of Abramov's formula, showing that if p is integrable then $h(T^p) = \int p\, d\mu \cdot h(T)$, thus restricting the class of aperiodic measure-preserving transformations that T can "integrably speed up" to look like. In recent work [BF],[BBF], Babichev, Burton, and Fieldsteel improve the Arnoux-Ornstein-Weiss result by demonstrating that p can be taken to be measurable with respect to a factor. More specifically, they consider a locally compact, second countable group G and a group extension of the form $T_\sigma : X \times G \to X \times G$ where $T_\sigma(x, g) = (Tx, \sigma(x)g)$ and $\sigma : X \times \mathbb{Z} \to G$ is a cocycle for T. They show that given any pair of aperiodic group extensions (by the same group G) where the first extension is ergodic, the first extension can be sped up to look like the second using a speedup function measurable with respect to the base factor X. In this sense, the work in [BBF] can be thought of as an extension of the results on relative orbit equivalence found in [F] and [G].

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It is then natural to ask what happens for higher dimensional actions. This first begs the question of what "speed up" means when there is no "up". One possibility is that the analogous function p be taken to be $\mathbf{p} : X \to \mathbb{N}^d$. In fact, we will show something more general:

Theorem 1.1. *Fix a locally compact, second countable group G and a neighborhood $U \subseteq G$ of the identity element of G. Let $(X, \mathcal{X}, \mu, \mathbf{T})$ and $(Y, \mathcal{Y}, \nu, \mathbf{S})$ be measure-preserving $\mathbb{Z}^d-$actions with $(Y, \mathcal{Y}, \nu, \mathbf{S})$ aperiodic. Let $\mathbf{T}^\sigma$ be an ergodic $G-$extension of $\mathbf{T}$ and $\mathbf{S}^\sigma$ be a $G-$extension of $\mathbf{S}$. Let $\mathbf{C} \subseteq \mathbb{Z}^d$ be any cone.*

Then there is a speedup $\overline{\mathbf{T}}^{\overline{\sigma}}$ of $\mathbf{T}^\sigma$ for which the speedup function is measurable with respect to $\mathcal{X}$ and takes values only in $\mathbf{C}$, such that $\overline{\mathbf{T}}^{\overline{\sigma}}$ is isomorphic to $\mathbf{S}^\sigma$, via an isomorphism of the form $(x, g) \mapsto (\phi(x), \overline{\alpha}(x)g)$, where $\phi : X \to Y$ is an isomorphism from a speedup of $\mathbf{T}$ to $\mathbf{S}$ and $\overline{\alpha} : X \to G$ is a measurable function taking values in U almost surely.

The idea of the proof is as follows: we approximate the action $\mathbf{S}$ by a sequence of partially-defined actions defined on larger and larger unions of Rohklin towers of $(Y, \mathcal{Y}, \nu, \mathbf{S})$. For each of these partially-defined actions, we first choose sets in X, the phase space of the $\mathbf{T}-$action, to mimic the sets found in the Rohklin towers. We next use a "quilting" argument to show that these sets can be realized as the orbit of a partially defined speedup of $\mathbf{T}$, with the speedup constructed at each step extending the speedup from the previous step. We show further that this can be done in such a way that respects the cocycles defining $\mathbf{T}^\sigma$ and $\mathbf{S}^\sigma$.

These types of constructions have their roots in the proof given in [AOW] and in the proof of Dye's Theorem [D1, D2] given by Hajian, Ito and Kakutani in [HIK]. Theorem 1.1 then yields a generalization of the main result of [BBF]: in fact, their result is exactly Theorem 1.1 with $d = 1$. While our proof follows the ideas of those in [BBF], it is not enough to simply use that result on each generator of the d-dimensional action, as the resulting speedups would not necessarily commute.

The next section provides the necessary definitions and background results. In Section 3 we develop a series of technical lemmas which will let us "quilt" together a collection of sets to yield a partially defined speedup of $\mathbf{T}$. The final section is then the recursive argument needed to yield the required speedup.

Note that if G is trivial, Theorem 1.1 can simplify to the following:

Corollary 1.2. *Let $(X, \mathcal{X}, \mu, \mathbf{T})$ and $(Y, \mathcal{Y}, \nu, \mathbf{S})$ be two ergodic $\mathbb{Z}^d-$actions and $\mathbf{C} \subseteq \mathbb{Z}^d$ be any cone. Then there exists a speedup $\overline{\mathbf{T}}$ of $\mathbf{T}$ such that $\overline{\mathbf{T}}$ is isomorphic to $\mathbf{S}$ and for which the speedup function takes values only in $\mathbf{C}$.*

Thus we have a generalization of Theorem 4 in [AOW] to higher dimensional actions.

2. Preliminaries

2.1. Background on group extensions.

2.1.1. *$\mathbb{Z}^d$-actions.* Let X be a Lebesgue probability space with measure μ. Given d commuting, invertible, measurable, measure-preserving transformations $T_1, T_2, ..., T_d$ of X, the collection $\{T_j\}$ generate a **$\mathbb{Z}^d$-action $\mathbf{T}$** on X. In particular, given vector $\mathbf{v} = (v_1, ..., v_d) \in \mathbb{Z}^d$ we write $\mathbf{T_v}$ for the transformation $T_1^{v_1} \circ T_2^{v_2} \circ ... \circ T_d^{v_d} : X \to X$. The action is said to be **ergodic** if the only sets invariant under every $\mathbf{T_v}, \mathbf{v} \in \mathbb{Z}^d$, are of zero or full measure.

2.1.2. *Group extensions.* Let G be a locally compact, second countable group; let λ be Haar measure on G (λ need not be finite). Given a $\mathbb{Z}^d$–action $\mathbf{T}$, a **cocycle** for $\mathbf{T}$ is a measurable function $\sigma : X \times \mathbb{Z}^d \to G$ satisfying the **cocycle equation**: for any $x \in X$ and $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^d$,

$$\sigma(x, \mathbf{v} + \mathbf{w}) = \sigma(\mathbf{T}_{\mathbf{v}}(x), \mathbf{w})\sigma(x, \mathbf{v}). \tag{2.1}$$

Given a cocycle σ for $\mathbf{T}$, we define a $\mathbb{Z}^d$–action $\mathbf{T}^\sigma$ on $X \times G$ by setting

$$\mathbf{T}^\sigma_{\mathbf{v}}(x, g) = (\mathbf{T}_{\mathbf{v}}(x), \sigma(x, \mathbf{v})g)$$

for each $\mathbf{v} \in \mathbf{Z}^d$. $\mathbf{T}^\sigma$ preserves $\mu \times \lambda$ and is called a **G-extension** of $\mathbf{T}$; conversely $\mathbf{T}$ is referred to as the **base** or **base factor** of $\mathbf{T}^\sigma$. In fact, a locally compact, second-countable group G admits an ergodic G–extension if and only if G is amenable [H],[Z].

In this setting, we can also define a cocycle on the orbit relation of $\mathbf{T}$, which is again labelled σ: if $z = \mathbf{T}_{\mathbf{v}}(x)$ for some $\mathbf{v} \in \mathbb{Z}^d$, we set $\sigma(x, z) = \sigma(x, \mathbf{v})$.

In this paper we use the symbol σ to refer to all our cocycles and when necessary, distinguish between the cocycles for different actions with subscripts (i.e. $\sigma_{\mathbf{T}}$ is the cocycle associated to the G–extension of $\mathbf{T}$).

2.1.3. *Factor maps and G–isomorphisms.* Let $(X, \mathcal{X}, \mu, \mathbf{T})$ and $(Y, \mathcal{Y}, \nu, \mathbf{S})$ be two measure-preserving $\mathbb{Z}^d$–actions with respective G–extensions $\mathbf{T}^\sigma$ and $\mathbf{S}^\sigma$. We say $\mathbf{S}^\sigma$ is a G–**factor** of $\mathbf{T}^\sigma$ if there is a measurable and measure-preserving map (defined on an invariant set of full measure, mapping onto an invariant set of full measure) $\Phi : X \times G \to Y \times G$ satisfying $\mathbf{S}^\sigma \circ \Phi = \Phi \circ \mathbf{T}^\sigma$ which is measurable with respect to the base factors, i.e. for all measurable $B \in \mathcal{Y}$, $\Phi^{-1}(B \times G) = A \times G$ a.s. for some $A \in \mathcal{X}$. Equivalently, this means

$$\Phi(x, g) = (\phi(x), \alpha(x)g)$$

where ϕ is a factor map from $(X, \mathcal{X}, \mu, \mathbf{T})$ to $(Y, \mathcal{Y}, \nu, \mathbf{S})$ and $\alpha : X \to G$ is measurable. If a G–factor map Φ exists which is almost surely $1-1$, we say $\mathbf{T}^\sigma$ and $\mathbf{S}^\sigma$ are G–**isomorphic** and we call Φ a G–isomorphism.

To say that two G–extensions are G–isomorphic means that the base transformations are isomorphic and the corresponding cocycles $\sigma_{\mathbf{S}}$ and $\sigma_{\mathbf{T}}$ on the orbit relations of the base transformations are cohomologous. The α in the previous paragraph is called the **transfer function** relating the cocycles. In particular, if $\mathbf{T}^\sigma$ is G–isomorphic to $\mathbf{S}^\sigma$ by the map Φ described above, then the cocycle $\sigma_{\mathbf{S}}$ must satisfy

$$\sigma_{\mathbf{S}}(\phi(x), \mathbf{v}) = \alpha(\mathbf{T}_{\mathbf{v}}x)\sigma_{\mathbf{T}}(x, \mathbf{v})\alpha(x)^{-1}.$$

Motivated by this fact, if $\mathbf{T}^\sigma$ is a G–extension of a $\mathbb{Z}^d$ action and $\alpha : X \to G$ is any measurable function, we define the **skewing of** σ **by** α to be the cocycle

$$\sigma^\alpha(x, \mathbf{v}) = \alpha(\mathbf{T}_{\mathbf{v}}x)\sigma(x, \mathbf{v})\alpha(x)^{-1}$$

and remark that $\mathbf{T}^\sigma$ is G–isomorphic to $\mathbf{T}^{\sigma^\alpha}$ by the map $(x, g) \mapsto (x, \alpha(x)g)$.

2.2. **Partial iterates and $\mathbb{Z}^d$−speedups.** We define a *filled cone* $\mathbf{C}$ to be any open, connected subset of $\mathbb{R}^d$ whose boundary consists of d distinct hyperplanes passing through the origin. A **cone** is the intersection of a filled cone with $(\mathbb{Z}^d - \{\mathbf{0}\})$. In particular, notice the zero vector does not belong to any cone. Given a cone $\mathbf{C}$ and any vector $\mathbf{v} \in \mathbb{Z}^d$, set $\mathbf{C_v} = \mathbf{C} \cap (\mathbf{C} + \mathbf{v})$.

Definition 2.1. *Given a $\mathbb{Z}^d-$action $(X, \mathcal{X}, \mu, \mathbf{T})$ and a set $Dom(R) \in \mathcal{X}$ of positive measure, a* **partial iterate** *of* $\mathbf{T}$ *is a $1-1$, measurable and measure-preserving function $R : Dom(R) \to X$ such that $R(x) = \mathbf{T}_{\mathbf{k}(x)}(x)$ for some measurable function $\mathbf{k} : Dom(R) \to \mathbb{Z}^d$. The function $\mathbf{k}$ is called the* **iterate function** *(of R). If $\mathbf{k}$ takes values only in some cone $\mathbf{C}$, we call R a* $\mathbf{C}-$**partial iterate**. *A $\mathbf{C}-$partial iterate R with $Dom(R) = X$ a.s. is called a* $\mathbf{C}-$**iterate** *of* $\mathbf{T}$.

Remark: In what follows, we will frequently say that a partial iterate R "**takes A to B**", where A and B are measurable subsets of X. What we mean by this phrase is that R is a partial iterate whose domain $Dom(R)$ is equal to A almost surely, and whose codomain is equal to B almost surely.

Definition 2.2. *Given two $\mathbb{Z}^d-$actions $(X, \mathcal{X}, \mu, \mathbf{T})$ and $(X, \mathcal{X}, \mu, \overline{\mathbf{T}})$, and given a cone $\mathbf{C}$, we say $\overline{\mathbf{T}}$ is a* $\mathbf{C}-$**speedup** *(or just* **speedup***) of* $\mathbf{T}$ *if there are $\mathbf{C}-$iterates $\overline{T}_1, \overline{T}_2, ..., \overline{T}_d$ of $\mathbf{T}$ such that $\overline{T}_i \circ \overline{T}_j = \overline{T}_j \circ \overline{T}_i$ for all i, j and such that for almost every $x \in X$ and every $\mathbf{v} = (v_1, ..., v_d) \in \mathbb{Z}^d$,*

$$\overline{\mathbf{T}}_{\mathbf{v}}(x) = \overline{T}_1^{v_1} \circ \overline{T}_2^{v_2} \circ \cdots \circ \overline{T}_d^{v_d}(x).$$

We use the word "speedup" here because this definition generalizes to $\mathbb{Z}^d$-actions the notion of "speedup" defined in [BBF]. In particular, when $d = 1$, there are only two cones, namely $\mathbf{C}_+ = \{1, 2, 3, ...\}$ and $\mathbf{C}_- = \{..., -3, -2, -1\}$. The main theorem of [BBF] is exactly our Theorem 1.1 with $d = 1, \mathbf{C} = \mathbf{C}_+$. That said, our usage of the word "speedup" in the context of $\mathbb{Z}^d-$actions is a bit of a misnomer, in that our speedups need not have any direct interpretation as systems which send "points forward in time more quickly than the original system".

Equivalently, $\overline{\mathbf{T}}$ is a $\mathbf{C}-$speedup of $\mathbf{T}$ if there is a measurable map $\mathbf{V} = (\mathbf{v}_1, ..., \mathbf{v}_d) : X \to \mathbf{C}^d$ such that the iterates $\mathbf{T}_{\mathbf{v}_1}, ..., \mathbf{T}_{\mathbf{v}_d}$ commute and the generators of $\overline{\mathbf{T}}$ are $\mathbf{T}_{\mathbf{v}_1}, ..., \mathbf{T}_{\mathbf{v}_d}$. $\mathbf{V}$ is called the **speedup function** of $\overline{\mathbf{T}}$.

2.3. **Speedup blocks, Rohklin towers and castles.**

2.3.1. *Speedup blocks.* We begin with some notation: given a nonnegative real number v_j, define $[v_j] = [0, v_j) \cap \mathbb{Z}$. Given a vector $\mathbf{v} = (v_1, ..., v_d)$ where $v_j > 0$ for all j, define $[\mathbf{v}] = \times_{j=1}^d [v_j]$. Thus $[\mathbf{v}]$ is a rectangle with side lengths v_j. We denote the cardinality of $[\mathbf{v}]$ by $|\mathbf{v}| = v_1 \cdot v_2 \cdots v_d$. Given $\mathbf{v}$ and $\mathbf{w} \in \mathbb{R}^d$, we say $\mathbf{v} \geq \mathbf{w}$ if $v_j \geq w_j$ for all j and we say $\mathbf{v} > \mathbf{w}$ if $v_j > w_j$ for all j. Given integer vectors $\mathbf{v} \leq \mathbf{w}$, set $[\mathbf{v}, \mathbf{w}] = \times_{j=1}^d ([v_j, w_j) \cap \mathbb{Z})$. Let $\{\mathbf{e}_1, ..., \mathbf{e}_d\}$ be the standard basis of $\mathbb{R}^d$ (just as well, $\mathbb{Z}^d$). Let $\mathbf{0} = (0, 0, ..., 0) \in \mathbb{Z}^d$ and let $\mathbf{1} = (1, 1, 1, ..., 1) \in \mathbb{Z}^d$. Given any set $S \subseteq \mathbb{Z}^d$, set $b_j(S) = \{\mathbf{v} \in S : \mathbf{v} + \mathbf{e}_j \in S\}$.

Definition 2.3. *Given a measure space $(X, \mathcal{X}, \mu)$ and $\mathbf{m} \in \mathbb{Z}^d$ with $\mathbf{m} \geq \mathbf{0}$, a* **rectangular collection** *of size $\mathbf{m}$ is a collection $\{A_\mathbf{v}\}_{\mathbf{v} \in [\mathbf{m}]}$ of subsets $A_\mathbf{v} \in \mathcal{X}$, where the sets are pairwise disjoint and all have the same measure.*

We remark that if at least one component of $\mathbf{m}$ is zero, then we obtain the empty collection of no sets; this is a rectangular collection.

Definition 2.4. *Let* $(X, \mathcal{X}, \mu, \mathbf{T})$ *be a* $\mathbb{Z}^d-$*action and let* $\{A_\mathbf{v}\}_{\mathbf{v}\in[\mathbf{m}]}$ *be a rectangular collection. A collection* $\overline{\mathbf{T}} = (\overline{T}_1, ..., \overline{T}_d)$ *of maps is called a* **partial speedup** *of* $\mathbf{T}$ *if*

(1) *each* $\overline{T}_j$ *is a partial iterate of* $\mathbf{T}$ *taking* $A_\mathbf{v}$ *to* $A_{\mathbf{v}+\mathbf{e}_j}$ *for all* $\mathbf{v} \in b_j([\mathbf{m}])$*;*
(2) *the* $\overline{T}_j$ *commute, i.e. for any* $\mathbf{v} \in b_j([\mathbf{m}]) \bigcap b_k([\mathbf{m}])$,
$$\overline{T}_j \circ \overline{T}_k(x) = \overline{T}_k \circ \overline{T}_j(x)$$
on $A_\mathbf{v}$.

In this setting the rectangular collection $\{A_\mathbf{v}\}_{\mathbf{v}\in[\mathbf{m}]}$ *is called a* **speedup block** *for* $\overline{\mathbf{T}}$*. If* $\mathbf{C}$ *is some cone such that for each* j*, the iterate function of* $\overline{T}_j$ *takes values only in* $\mathbf{C}$*, we say* $\overline{\mathbf{T}}$ *is a* $\mathbf{C}-$**partial speedup**.

2.3.2. *Rohklin towers.*

Definition 2.5. *Let* $\mathbf{m} > \mathbf{0}$*. A* **Rohklin tower** τ *for a* $\mathbb{Z}^d-$*action* $(X, \mathcal{X}, \mu, \mathbf{T})$ *is a rectangular collection* $\{A_\mathbf{v}\}_{\mathbf{v}\in[\mathbf{m}]}$ *of measurable sets such that* $T_j(A_\mathbf{v}) = A_{\mathbf{v}+\mathbf{e}_j}$ *for all* $\mathbf{v} \in b_j([\mathbf{m}])$*. Each set* $A_\mathbf{v}$ *is called a* **level** *of the tower;* $\mathbf{m}$ *is called the* **height** *of the tower;* $A_\mathbf{0}$ *is called the* **base** *of the tower, and the common value* $\mu(A_\mathbf{v})$ *is called the* **width** *of the tower.*

Speedup blocks are closely related to Rohklin towers: given a speedup $\overline{\mathbf{T}}$ of $\mathbf{T}$, any Rohklin tower for $\overline{\mathbf{T}}$ is a speedup block for the restriction of $\overline{\mathbf{T}}$ to the tower.

We see that for any Rohklin tower of height $\mathbf{m}$, $A_\mathbf{v} = \mathbf{T}_\mathbf{v}(A_\mathbf{0})$ for all $\mathbf{v} \in [\mathbf{m}]$. A **column** of a Rohklin tower is another tower of the form $\{\mathbf{T}_\mathbf{v}(B_\mathbf{0})\}_{\mathbf{v}\in[\mathbf{m}]}$ where $B_\mathbf{0}$ is a measurable subset of $A_\mathbf{0}$. We denote by $|\tau|$ the union of the levels of the tower, and let the **interior** of the tower be
$$\mathrm{int}(\tau) = \bigcup_{\mathbf{v}\in[\mathbf{1},\mathbf{m}-\mathbf{1}]} A_\mathbf{v}.$$
The **error set** of a Rohklin tower τ is $E(\tau) = X - \bigcup_{\mathbf{v}\in[\mathbf{m}]} A_\mathbf{v}$.

The classical Rohklin lemma for $\mathbb{Z}^d-$actions [OW] can be stated as follows:

Lemma 2.6 (Rohklin Tower Lemma)**.** *Let* $(Y, \mathcal{Y}, \nu, \mathbf{S})$ *be an aperiodic* $\mathbb{Z}^d-$*action. Then for every* $\epsilon > 0$ *and every integer vector* $\mathbf{m} > 0$*, there is a Rohklin tower* τ *for* $\mathbf{S}$ *of height* $\mathbf{m}$ *such that* $\nu(E(\tau)) = \epsilon$.

2.3.3. *Castles.* Given a $\mathbb{Z}^d-$action $(Y, \mathcal{Y}, \nu, \mathbf{S})$, a **castle** $\mathcal{C}$ is a finite collection $\tau_1, ..., \tau_s$ of Rohklin towers for $\mathbf{S}$, where $|\tau_i| \bigcap |\tau_j| = \emptyset$ for all $i \neq j$. Denote by $|\mathcal{C}|$ the union of all the levels of all the towers comprising the castle, and let the **interior** of the castle, denoted $\mathrm{int}(\mathcal{C})$, be the union of the interiors of the towers comprising $\mathcal{C}$. The **error set** of $\mathcal{C}$ is $E(\mathcal{C}) = X - |\mathcal{C}|$. By a **level** of $\mathcal{C}$, we mean a level of any of the towers comprising $\mathcal{C}$, and we define a **column** of $\mathcal{C}$ to be a column of any of the towers comprising $\mathcal{C}$. The set of levels of the castle $\mathcal{C}$ is denoted $L(\mathcal{C})$, and the $\sigma-$algebra generated by the levels of $\mathcal{C}$ is denoted $\mathcal{L}(\mathcal{C})$.

Given a tower τ of size $\mathbf{m}$ and a finite, measurable partition $\mathcal{Q} = \{Q_1, ..., Q_s\}$ of the base of τ, we obtain a castle $\tau_\mathcal{Q}$ whose bases are the atoms of $\mathcal{Q}$.

Given a finite measurable partition $\mathcal{Q} = \{Q_1, ..., Q_s\}$ of $|\tau|$ (just as well, of Y), we define the partition $\mathcal{Q}_\tau$ of the base of τ by setting the atoms of this partition to be maximal sets B_j such that for every $\mathbf{v} \in [\mathbf{m}]$, $\mathbf{S}_\mathbf{v}(B_j)$ is contained entirely within one atom of $\mathcal{Q}$. We call the partition $\mathcal{Q}_\tau$ the **partition into $\mathcal{Q}-$names** and the resulting castle $\tau_{(\mathcal{Q}_\tau)}$ the **castle of $\mathcal{Q}-$columns in τ.**

Given a castle $\mathcal{C}$ and a finite measurable partition $\mathcal{Q}$ of $|\mathcal{C}|$ (or of Y), by repeating the construction of the previous paragraph on each tower comprising $\mathcal{C}$, we obtain a castle $\mathcal{C}_\mathcal{Q}$ which we call the **castle of $\mathcal{Q}-$columns in $\mathcal{C}$.**

2.3.4. *Cutting and stacking constructions.* Following the work in [AOW], we make the following definition:

Definition 2.7. *Given two castles $\mathcal{C}_1$ and $\mathcal{C}_2$ for $(Y, \mathcal{Y}, \nu, \mathbf{S})$, we say $\mathcal{C}_2$ is* **obtained from $\mathcal{C}_1$ via a cutting and stacking construction** *if:*

1. $|\mathcal{C}_1| \subseteq \text{int}(\mathcal{C}_2)$*;*
2. *there is a finite partition $\mathcal{Q}$ of the bases of $\mathcal{C}_1$ such that each level of the castle $(\mathcal{C}_1)_\mathcal{Q}$ is a level of $\mathcal{C}_2$; and*
3. *for each tower of $(\mathcal{C}_1)_\mathcal{Q}$, there is a tower of $\mathcal{C}_2$ that contains it.*

Observe that criterion (1) above implies that if $\{A_\mathbf{v}\}_{\mathbf{v}\in[\mathbf{m}]}$ is a tower in $\mathcal{C}_2$ and if $A_\mathbf{w}$ is a base of a tower of $(\mathcal{C}_1)_\mathcal{Q}$ of size $\mathbf{h}$, then we have $\mathbf{w} + \mathbf{h} < \mathbf{m}$.

Lemma 2.8 (Castle Lemma)**.** *Let G be a locally compact, second countable group and let $\mathbf{S}^\sigma$ be an ergodic $G-$extension of the $\mathbb{Z}^d-$action $(Y, \mathcal{Y}, \nu, \mathbf{S})$. Let $\{U_k\}_{k=1}^\infty$ be a neighborhood base for G at e_G. Then there is a sequence $\{\mathcal{C}_k\}_{k=1}^\infty$ of castles for $\mathbf{S}$ satisfying:*

1. *for each k, all towers in the castle $\mathcal{C}_k$ have the same height $\mathbf{N}_k$;*
2. *for each k, $\mathcal{C}_{k+1}$ is obtained from $\mathcal{C}_k$ via a cutting and stacking construction;*
3. $\nu\left(\bigcup_{k=1}^\infty \mathcal{C}_k\right) = 1$*;*
4. $\bigcup_{k=1}^\infty \mathcal{L}(\mathcal{C}_k) = \mathcal{Y}$*; and*
5. *for each tower τ in $\mathcal{C}_k$, and for each $\mathbf{v} \in [\mathbf{N}_k]$, there is a group element $g \in G$ such that for all y in the base of τ, $\sigma(y, \mathbf{v}) \in U_k g$.*

Proof. The proof is divided into two phases: first, following the work in [AOW] for $\mathbb{Z}-$actions, we construct a sequence of towers via cutting and stacking constructions. Second, we modify these towers in a way similar to [BBF] to yield a sequence of castles satisfying the conclusions of the lemma.

Phase 1: Construction of the towers $\{\tau_i\}$. Let $\{\epsilon_i\}_{i=1}^\infty$ be a decreasing sequence of positive numbers such that $\sum_{i=1}^\infty \epsilon_i < 1/2$, and choose a sequence $\{\mathbf{N}_i\}_{i=1}^\infty$ of vectors in $\mathbb{Z}^d$, where $\mathbf{N}_i = (N_i(1), ..., N_i(d)\,)$, such that

$$\frac{2\sum_{j=1}^d \left(N_i(j)\, \prod_{k\neq j} N_{i-1}(k)\right)}{|\mathbf{N}_i|} < \frac{\epsilon_i}{4}. \tag{2.2}$$

For each i, define the **boundary** of $[\mathbf{N}_i]$ to be the set of indices $\mathbf{v} \in [\mathbf{N}_i]$ such that $\mathbf{v} \pm \mathbf{e}_j \notin [\mathbf{N}_i]$ for some $j \in \{1, ..., d\}$. Next, we define the **collar** of $[\mathbf{N}_i]$ to be the following set of indices in $[\mathbf{N}_i]$:

$$\{\mathbf{v} \in [\mathbf{N}_i] : \mathbf{v} \pm N_{i-1}(k)\, \mathbf{e}_k \notin [\mathbf{N}_i] \text{ for some } k = 1, ..., d\}.$$

In other words, the collar of $[\mathbf{N}_i]$ is the set of indices which are close to its boundary, where "close" is defined by the size of $[\mathbf{N}_{i-1}]$. Note that the portion of $[\mathbf{N}_i]$ contained in its collar is bounded by the left-hand expression in inequality (2.2) above.

Next, take a sequence of Rohklin towers τ_i^1 of size $\mathbf{N}_i$, whose error sets have measure $\frac{1}{2}\epsilon_i$; let B_i^1 be the base of each τ_i^1. Define the **boundary** of each tower to be $\{\mathbf{S}_\mathbf{v}(B_i^1) : \mathbf{v}$ is in the boundary of $[\mathbf{N}_i]\}$.

In order to satisfy criteria (2) of the lemma, we successively alter the towers by an inductive process. The idea is to remove points from the i^{th} tower which are not in the interior of the $(i+1)^{st}$ tower, and then to justify that the resulting tower has measure only slightly smaller than the original. Now for the details: our first step is to

(i) remove y from B_1^1 if for any $\mathbf{v} \in [\mathbf{N}_1]$, $\mathbf{S}_\mathbf{v}(y)$ is in the boundary of τ_2^1 (notice that the measure of this set is bounded by the measure of the set of points in τ_2^1 in the collar of $[\mathbf{N}_2]$, which by (2.2) is less than $\frac{\epsilon_2}{4}$), and
(ii) then remove from B_1^1 any point y such that for any $\mathbf{v} \in [\mathbf{N}_1]$, $\mathbf{S}_\mathbf{v} y \in E(\tau_2^1)$ (having already removed such points in the collar of $[\mathbf{N}_2]$, the only points in $\tau_1^1 \cap E(\tau_2^1)$ are those for which $\mathbf{S}_\mathbf{v} y \in E(\tau_2^1)$ for every $\mathbf{v} \in [\mathbf{N}_1]$; this set of points is bounded in measure by $\nu(E(\tau_2^1)) = \frac{\epsilon_2}{2}$).

Let B_1^2 be set of the points remaining in B_1^1 after these two steps. Let $\tau_1^2 = \{\mathbf{S}_\mathbf{v} B_1^2\}_{\mathbf{v}\in[\mathbf{N}_1]}$ (this tower is our first modification of τ_1^1). Notice

$$\nu(\tau_1^2) \geq \nu(\tau_1^1) - \left(\frac{\epsilon_2}{4} + \frac{\epsilon_2}{2}\right) > \nu(\tau_1^1) - \epsilon_2,$$

and that $\nu(E(\tau_1^2)) < \mu(E(\tau_1^1)) + \epsilon_2$.

For our second step, we similarly remove the following points from B_2^1:

(i) those associated to points in the tower τ_2^1 which intersect the collar of τ_3^1.
(ii) those associated to points in τ_2^1 which intersect the error set $E(\tau_3^1)$.

We define B_2^2 analogously and let $\tau_2^2 = \{\mathbf{S}_\mathbf{v} B_2^2\}_{\mathbf{v}\in[\mathbf{N}_2]}$. Similarly to our first step, we will have $\nu(\tau_2^2) \geq \nu(\tau_2^1) - \epsilon_3$, and $\nu(E(\tau_2^2)) < \nu(E_2^1) + \epsilon_3$.

This in turn means we must modify τ_1^2, removing those points which intersect $E(\tau_2^2)$ (which is larger than $E(\tau_2^1)$). Much like how we previously removed points in τ_1^1 that intersected $E(\tau_2^1)$, we see that the points we must remove at this step have measure at most $\nu(E(\tau_2^2)) - \nu(E(\tau_2^1)) \leq \epsilon_3$. Thus we create τ_1^3, our second modification of the first tower, and we note $\nu(\tau_1^3) > \nu(\tau_1^1) - \epsilon_2 - \epsilon_3$.

We continue in this manner. At the i^{th} step, we modify τ_i^1 by removing points from B_i^1, then let $\tau_i^2 = \{\mathbf{S}_\mathbf{v} B_i^2\}_{\mathbf{v}\in[\mathbf{N}_i]}$ and note $\nu(\tau_i^2) \geq \nu(\tau_i^1) - \epsilon_{i+1}$. The corresponding error set $E(\tau_i^2)$ has measure which is less than $\nu(E(\tau_i^1)) + \epsilon_{i+1}$. We then modify all the previous towers to compensate for the increased error set; this results in the removal of a portion of those towers which has measure at most ϵ_{i+1}.

At the end of the i^{th} step, we will have defined $\{\tau_h^{i+2-h}\}$ for $h \in \{1, ..., i\}$, with

$$\nu(\tau_h^{i+2-h}) \geq \nu(\tau_h^1) - \epsilon_{h+1} - ... - \epsilon_{i+1}.$$

Defining $\tau_i = \cap_{j=1}^\infty \tau_i^j$, we end up with a sequence $\{\tau_i\}$ of towers with $\lim_{i\to\infty}\nu(\tau_i) = 1$. Let B_i be the base of tower τ_i.

Phase 2: Altering the towers and building the castles. We again successively alter the towers by an inductive process. To start, fix a sequence of finite partitions $\{\mathcal{P}_k\}$ which generate $\mathcal{Y}$ and a sequence $\{\alpha_k\}_{k=1}^{\infty}$ of positive numbers such that $\sum_k \alpha_k < 1$. As G is locally compact, we can choose compact $K_1 \subset G$ so that if

$$B_1' = \bigcap_{\mathbf{v}\in[\mathbf{N}_1]} \{y \in B_1 : \sigma(y, \mathbf{v}) \in K_1\},$$

then $\nu(B_1') > (1 - \alpha_1)\nu(B_1)$. Let τ_1' be the portion of τ_1 over B_1', i.e. $\tau_1' = \{\mathbf{S}_\mathbf{v} B_1'\}_{\mathbf{v}\in[\mathbf{N}_1]}$. We partition K_1 into sets $\{K_{1,i}\}_{i=1}^{s_1}$ such that for each i, there exists a $g_{1,i} \in G$ with $K_{1,i} \subset U_1\, g_{1,i}$. Let $\kappa_1 : G \to G$ be given by

$$\kappa_1(g) = \begin{cases} g_{1,i} & \text{if } g \in K_{1,i} \\ e_G & \text{otherwise.} \end{cases}$$

We next partition B_1' according to both the values of $\{\kappa_1(\sigma(y, \mathbf{v}))\}_{\mathbf{v}\in[\mathbf{N}_1]}$ and $\{\mathcal{P}_1(T_\mathbf{v} y)\}_{\mathbf{v}\in[\mathbf{N}_1]}$. Calling this partition $\mathcal{Q}_1$, we let $\mathcal{C}_1' = (\tau_1)_{\mathcal{Q}_1}$.

Now, choose compact $K_2 \subset G$ so that if

$$B_2' = \bigcap_{\mathbf{v}\in[\mathbf{N}_2]} \{y \in B_2 : \sigma(y, \mathbf{v}) \in K_2\}$$

then $\nu(B_2') > (1 - \alpha_2)\nu(B_2)$. Let τ_2' be the portion of τ_2 over B_2'. We can partition K_2 into sets $\{K_{2,i}\}_{i=1}^{s_2}$ such that for each i, there exists a $g_{2,i} \in G$ with $K_{2,i} \subset U_1\, g_{2,i}$. Let $\kappa_2 : G \to G$ be given by

$$\kappa_2(g) = \begin{cases} g_{2,i} & \text{if } g \in K_{2,i} \\ e_G & \text{otherwise.} \end{cases}$$

Define $\mathcal{R}_1$ to be the partition of Y into the levels of $\mathcal{C}_1'$. Now partition B_2' according to the values of $\{\kappa_2(\sigma(y, \mathbf{v}))\}_{\mathbf{v}\in[\mathbf{N}_2]}$, $\{\mathcal{P}_2(T_\mathbf{v} y)\}_{\mathbf{v}\in[\mathbf{N}_2]}$, and $\{\mathcal{R}_1(T_\mathbf{v} y)\}_{\mathbf{v}\in[\mathbf{N}_2]}$. Calling this partition $\mathcal{Q}_2$, we set $\mathcal{C}_2' = (\tau_2)_{\mathcal{Q}_2}$. In particular, the towers comprising $\mathcal{C}_2'$ each have a fixed pattern of locations of the $\mathcal{C}_1'$ towers and fixed $(\mathcal{P}_2 \vee \kappa_2) - \mathbf{N}_2$-names.

Note that to maintain conclusion (2) of the lemma, we must remove points from $\mathcal{C}_1'$ that intersect with the new error portion of $\mathcal{C}_2'$. But the set of such points has measure less than α_2.

We continue in the same way, constructing $\mathcal{C}_k'$ and altering the proceeding $\mathcal{C}_i'$'s at each step. The resulting castles will be denoted $\mathcal{C}_k$. By construction, conclusions (1) , (2) and (5) of the lemma hold. Since the partitions $\mathcal{P}_k$ generate $\mathcal{Y}$, we have (4). Last, note that $\nu(\mathcal{C}_k) > \nu(\mathcal{C}_k') - \sum_{i=k+1}^{\infty} \alpha_i$, and by our choice of α_k's, this yields conclusion (3). □

3. Quilting arguments

The goal of this section is to prove the following theorem, which is central to a recursive argument used in the proof of Theorem 1.1. Colloquially, the theorem says that if we consider a rectangular collection of size $\mathbf{N}$ and a collection of (disjoint) smaller speed up blocks of size $\mathbf{h}$ which sit inside the rectangular collection, then we can define a partial speedup which has the rectangular collection as a speedup block and which extends the previously defined partial speedups.

Theorem 3.1. *Fix $\mathbf{N} \geq \mathbf{0}$, $\mathbf{h} \geq \mathbf{0}$, and a cone $\mathbf{C} \subset \mathbb{Z}^d$. Let $\mathbf{T}^\sigma$ be an ergodic $G-$extension of the $\mathbb{Z}^d-$action $(X, \mathcal{X}, \mu, \mathbf{T})$ and let $\{A_\mathbf{v}\}_{\mathbf{v}\in[\mathbf{N}]}$ be a rectangular collection of subsets of X. Let U be a neighborhood of e_G, and suppose that for each $\mathbf{v} \in [\mathbf{N}]$, we are given a measurable function $g_\mathbf{v} : A_\mathbf{0} \to G$ (where $g_\mathbf{0}(x)$ is the constant function e_G).*

Suppose that there are vectors $\mathbf{k}_1, ..., \mathbf{k}_r \in [\mathbf{N}-\mathbf{h}]$ such that the sets $\bigcup_{\mathbf{v}\in[\mathbf{h}]} A_{\mathbf{k}_j+\mathbf{v}}$ are pairwise disjoint, and that each $\{A_{\mathbf{k}_j+\mathbf{v}}\}_{\mathbf{v}\in[\mathbf{h}]}$ is a speedup block for a $\mathbf{C}-$partial speedup $\overline{\mathbf{T}}_j$ of $\mathbf{T}$.

Then, there is a $\mathbf{C}-$partial speedup $\overline{\mathbf{T}}$ extending the $\overline{\mathbf{T}}_j$ such that $\{A_\mathbf{v}\}_{\mathbf{v}\in[\mathbf{N}]}$ is a speedup block for $\overline{\mathbf{T}}$, and for all $\mathbf{v}$ in

$$[\mathbf{N}] - \bigcup_{j=1}^{r} \bigcup_{\mathbf{w}\in[\mathbf{h}],\mathbf{w}\neq\mathbf{0}} \{\mathbf{k}_j + \mathbf{w}\},$$

we have $\sigma_\mathbf{T}(x, \overline{\mathbf{T}}_\mathbf{v}(x))(g_\mathbf{v}(x))^{-1} \in U$ for a.e. $x \in A_\mathbf{0}$.

We prove this theorem via a series of technical lemmas, which describe how increasingly complicated configurations of sets and partial iterates can be "quilted" together to form a speedup block for a partial speedup of $\mathbf{T}$.

3.1. **Initial arguments.**

The goal of this subsection is to prove Lemma 3.9, which essentially says that if we are given a rectangular collection of sets and partial iterates defined on a "lower triangular" subset of the rectangular collection, then we can "complete" the rest of the rectangle, i.e. we can define partial iterates on the remainder of the rectangular collection so that the rectangular collection becomes a speedup block for a partial speedup of $\mathbf{T}$.

We begin by showing that given two subsets of X, we can find an iterate of the action $\mathbf{T}$ that sends a portion of one set to the other, and in such a way that the cocycle lies in a predetermined subset of G.

Lemma 3.2. *Fix a cone $\mathbf{C}$ and suppose $\mathbf{T}^\sigma$ is an ergodic $G-$extension of the $\mathbb{Z}^d-$action $(X, \mathcal{X}, \mu, \mathbf{T})$. For all sets $A, B \subseteq X$ of positive measure, for all $\mathbf{v} \geq \mathbf{0}$, and for any non-empty open set $U \subseteq G$, there is a set $A' \subseteq A$ and a vector $\mathbf{n} \in \mathbf{C}_\mathbf{v}$ such that*

1. *$\mu(A') > 0$;*
2. *$\mathbf{T}_\mathbf{n}(A') \subseteq B$; and*
3. *$\sigma(x, \mathbf{n}) \in U$ for all $x \in A'$.*

Proof. Given A, B and U, choose non-empty open subsets V_0 and V_1 of G so that $e_G \in V_0$ and $V_1 V_0^{-1} \subseteq U$. Since $\mathbf{C}$ is a cone, there exists a Følner sequence $\{F_n\}$ for the group $\mathbb{Z}^d$ consisting of parallepipeds, each of whom are subsets of $\mathbf{C}_\mathbf{v}$. Without loss of generality, assume this sequence is tempered (see Proposition 1.4 of [L]). Now, applying the pointwise ergodic theorem of [L] to the indicator function of $B \times V_1$, we can conclude that for almost every $(x, g) \in A \times V_0$, there exists (infinitely many) $\mathbf{m} \in \mathbf{C}_\mathbf{v}$ such that $(\mathbf{T}^\sigma)_\mathbf{m}(x, g) \in B \times V_1$.

Hence there is a $g_0 \in V_0$ such that for almost every $x \in A$, there is $\mathbf{m} \in \mathbf{C_v}$ such that $(\mathbf{T}^\sigma)_\mathbf{m}(x, g_0) \in B \times V_1$. For each $\mathbf{m} \in \mathbf{C_v}$, let $A_\mathbf{m} = \{x \in A : (\mathbf{T}^\sigma)_\mathbf{m}(x, g_0) \in B \times V_1\}$. Since $A = \bigcup_\mathbf{m} A_\mathbf{m}$ almost surely, there exists $\mathbf{n} \in \mathbf{C_v}$ such that $\mu(A_\mathbf{n}) > 0$; set A' to be this $A_\mathbf{n}$. We have, for any $x' \in A'$, $\sigma(x', \mathbf{n})g_0 \in V_1$ so $\sigma(x', \mathbf{n}) \in V_1 g_0^{-1} \subset V_1 V_0^{-1} \subseteq U$ as desired. □

The next lemma says that if the two sets have the same measure, we can, by repeating the above procedure, construct a partial iterate that takes one set to the other, with the cocycle similarly well-behaved.

Lemma 3.3. *Fix a cone* $\mathbf{C}$ *and suppose* $\mathbf{T}^\sigma$ *is an ergodic* $G-$*extension of the* $\mathbb{Z}^d-$*action* $(X, \mathcal{X}, \mu, \mathbf{T})$. *Given two subsets* $A, B \subseteq X$ *of equal positive measure, then for all* $\mathbf{v} \in \mathbb{Z}^d$, *and for all non-empty open sets* $U \subseteq G$, *there is a partial iterate* R *of* $\mathbf{T}$ *such that:*

1. *R takes A to B;*
2. *the iterate function $\mathbf{k}$ of R takes values only in $\mathbf{C_v}$; and*
3. *for almost every $x \in A$, $\sigma(x, \mathbf{k}(x)) \in U$.*

Proof. Given A, B, U, and $\mathbf{v}$, fix some decreasing, positive sequence ϵ_j satisfying $\sum_j \epsilon_j < \infty$. Define

$$a_1 = \sup\{\mu(A') : A' \text{ satisfies the conclusions of Lemma 3.2 for } A, B, U, \text{ and } \mathbf{v}\}.$$

Choose A_1 to be a set satisfying the conclusions of Lemma 3.2 for A, B, U and $\mathbf{v}$ where $\mu(A_1) > a_1 - \epsilon_1$; let $\mathbf{n}_1$ be the corresponding vector coming from Lemma 3.2 such that $\mathbf{T}_{\mathbf{n}_1}(A_1) \subseteq B$.

If $\mu(A_1) = \mu(A)$, we are done (set $R = \mathbf{T}_{\mathbf{n}_1}$). Otherwise, set $A^1 = A - A_1$, $B^1 = B - \mathbf{T}_{\mathbf{n}_1}(A_1)$ and

$$a_2 = \sup\{\mu(A') : A' \text{ satisfies the conclusions of Lemma 3.2 for } A^1, B^1, U \text{ and } \mathbf{v}\}.$$

Then choose $A_2 \subseteq A^1$ such that A_2 satisfies the conclusions of Lemma 3.2 for A^1, B^1, U and $\mathbf{v}$ where $\mu(A_2) > a_2 - \epsilon_2$.

Continuing in this fashion, we obtain a pairwise disjoint sequence of sets $A_1, A_2, ...$ and corresponding vectors $\mathbf{n}_1, \mathbf{n}_2, ... \in \mathbf{C_v}$ such that the sets $\mathbf{T}_{\mathbf{n}_j}(A_j)$ are disjoint subsets of B.

If at any point, $\mu(\bigcup_{j=1}^p A_j) = \mu(A)$, we are done (define R so that its restriction to each A_j is $\mathbf{T}_{\mathbf{n}_j}$).

Otherwise, for all $p > 0$, $\mu\left(\bigcup_{j=1}^p A_j\right) < \mu(A)$. Suppose $\mu(\bigcup_{j=1}^\infty A_j) < \mu(A)$; then by Lemma 3.2, there is a set $A' \subseteq A - \bigcup_{j=1}^\infty A_j$ and a vector $\mathbf{n}'$ satisfying the conclusions of Lemma 3.2. However, since $\mu(A) < \infty$, $\sum_{j=1}^\infty \mu(A_j) < \infty$, so $\lim_{j\to\infty} \mu(A_j) = 0$ and also $\lim_{j\to\infty}(\mu(A_j) + \epsilon_j) = 0$. Therefore, for some j we have

$$a_j < \mu(A_j) + \epsilon_j < \mu(A')$$

which contradicts the choice of a_j. Therefore $\mu(\bigcup_{j=1}^\infty A_j) = \mu(A)$, and we can therefore define R on $\bigcup_{j=1}^\infty A_j$ by setting $R(x) = \mathbf{T}_{\mathbf{n}_j}(x)$ whenever $x \in A_j$. □

If we think of the last lemma as creating a "patch" between two sets, the next lemma tells us how we can add one patch onto another: we start with a partial iterate between two sets and construct another partial iterate that connects them to a third set.

Lemma 3.4. *Fix a cone* $\mathbf{C}$ *and suppose* $\mathbf{T}^\sigma$ *is an ergodic* $G-$*extension of the* $\mathbb{Z}^d-$*action* $(X, \mathcal{X}, \mu, \mathbf{T})$*. Given three subsets* $A, B, C \subseteq X$ *of equal positive measure and a partial iterate* R *of* $\mathbf{T}$ *taking* A *to* B*, then for all* $\mathbf{v} \in \mathbb{Z}^d$*, and for any non-empty open set* $U \subseteq G$*, there is a partial iterate* R' *of* $\mathbf{T}$ *such that:*

1. *R' takes B to C;*
2. *the iterate function $\mathbf{k}$ of R' takes values only in $\mathbf{C}_\mathbf{v}$; and*
3. *for almost every $x \in A$, $\sigma_\mathbf{T}(x, R' \circ R(x)) \in U$.*

Proof. Given U, choose open subsets $V_1, V_2, ...$ and $W_1, W_2, ...$ of G such that $W_j V_j \subseteq U$ for all j and $\bigcup_j V_j = G$. Partition A into measurable sets $A_1, A_2, ...$ where

$$A_j = \left\{ x \in A : \sigma_\mathbf{T}(x, R(x)) \in V_j - \bigcup_{i=1}^{j-1} V_i \right\}$$

and for each j, let $B_j = R(A_j)$. The sets B_j form a measurable partition of B. Partition C into measurable sets $C_1, C_2, ...$ so that $\mu(C_j) = \mu(B_j) = \mu(A_j)$ for all j. Use Lemma 3.3 to construct maps $R'_j : B_j \to C_j$ such that the iterate function of R'_j takes values only in $\mathbf{C}_\mathbf{v}$ and $\sigma_\mathbf{T}(z, R'_j(z)) \in W_j$ for almost every $z \in B_j$. Then define R' so that it coincides with R'_j on each B_j; we have for a.e. $x \in A_j$, $\sigma_\mathbf{T}(x, R' \circ R(x)) \in W_j V_j \subseteq U$ as desired. □

So far we have found partial iterates, i.e. 1-dimensional actions with particular properties. Now we move to the d-dimensional scenario: first, let Q denote the $d-$dimensional cube $\{0,1\}^d$. For each $j \in \{0, ..., d\}$, set $Q_j = \{\mathbf{v} \in Q : v_1 + ... + v_d = j\}$. Notice that Q_d consists of exactly 1 point, which we think of as the "last" corner of the cube. The next lemma says that if we have partial iterates defined on the parts of the cube which do not involve Q_d, then we can "finish the cube", i.e. create a d-dimensional action which is a $\mathbf{C}$-partial speedup extending the iterates already defined, whose speedup block is in the shape of Q.

Lemma 3.5. *Fix a cone* $\mathbf{C}$ *and suppose* $\mathbf{T}^\sigma$ *is an ergodic* $G-$*extension of the* $\mathbb{Z}^d-$*action* $(X, \mathcal{X}, \mu, \mathbf{T})$ *and let* U *be an open subset of* G*. Suppose further that* $\{A_\mathbf{y}\}_{\mathbf{y} \in Q}$ *is a rectangular collection of size* $(2, 2, ..., 2)$*, and*

1. *for every $\mathbf{y} \in b_j(Q)$ with $\mathbf{y} + \mathbf{e}_j \neq \mathbf{1}$, there is a $\mathbf{C}-$partial iterate I_j taking $A_\mathbf{y}$ to $A_{\mathbf{y}+\mathbf{e}_j}$; and*
2. *the partial iterates described in (1) commute, i.e. if $\mathbf{y} \in b_j(Q) \bigcap b_k(Q)$ is such that $\mathbf{y} + \mathbf{e}_j + \mathbf{e}_k \neq \mathbf{1}$, then $I_j \circ I_k = I_k \circ I_j$ a.s. on $A_\mathbf{y}$.*

Then there exists a $\mathbf{C}-$*partial speedup* $\overline{\mathbf{T}} = (\overline{T}_1, ..., \overline{T}_d)$ *of* $\mathbf{T}$ *such that*

- i) *$\overline{T}_j = I_j$ wherever the iterate I_j is defined;*
- ii) *$\{A_\mathbf{y}\}_{\mathbf{y} \in Q}$ is a speedup block for $\overline{\mathbf{T}}$; and*
- iii) *$\sigma_\mathbf{T}(x, \overline{T}_1 \overline{T}_2 \cdots \overline{T}_d(x)) \in U$ for a.e. $x \in A_\mathbf{0}$.*

Proof. Notice that for $\mathbf{x} \in Q_{d-1}$, exactly one component of $\mathbf{x}$ is zero, so we can enumerate the elements of Q_{d-1} by setting $\mathbf{g}_k$ to be the element of Q_{d-1} with k^{th} component 0. For each $j \in \{1, ..., d\}$, set

$$R_j = I_1 \circ I_2 \circ \cdots I_{j-1} \circ I_{j+1} \circ I_{j+2} \circ \cdots \circ I_d : A_\mathbf{0} \to A_{\mathbf{g}_j}.$$

Thus $R_j(x)$ has the form $\mathbf{T}_{\mathbf{r}_j(x)}(x)$.

Let $\mathcal{P}$ be the partition of $A_{\mathbf{0}}$ into maximal sets on which $\mathbf{r}_1, ...\mathbf{r}_d$ are constant; let the atoms of $\mathcal{P}$ be denoted $P_1, P_2, ...$ Partition $A_{\mathbf{1}}$ into sets $D_1, D_2, ...$ where $\mu(P_j) = \mu(D_j)$ for all j.

Consider an arbitrary partition element P_i. Note that for each j, $\mathbf{r}_j(x)$ is constant on P_i, thus we can denote it by $\mathbf{r}_j$. Let $\mathbf{v}$ be such that

$$\mathbf{C}_{\mathbf{v}} \subseteq \mathbf{C} \cap (\mathbf{C} + (\mathbf{r}_2 - \mathbf{r}_1)\,) \cap ... \cap (\mathbf{C} + (\mathbf{r}_d - \mathbf{r}_1)\,).$$

Use Lemma 3.4 (with sets $P_i, R_1(P_i)$, and D_i; partial iterate R_1 restricted to P_i; the vector $\mathbf{v}$ specified above, and the set U from the hypothesis) to construct a partial iterate $R_1' : R_1(P_i) \to D_i$ whose iterate function $\mathbf{k}_1$ takes values in $\mathbf{C}_{\mathbf{v}}$ (and thus in $\mathbf{C}$) and where $\sigma_{\mathbf{T}}(x, R_1' \circ R_1(x)) \in U$ for a.e. $x \in P_i$.

For $j > 1$, we define partial iterates R_j' as follows: for $z \in R_j(P_i)$, find $x \in P_i$ with $R_j(x) = z$. Then set $\mathbf{k}_j(z) = -\mathbf{r}_j + \mathbf{r}_1 + \mathbf{k}_1(R_1(x))$ and define $R_j'(z) = \mathbf{T}_{\mathbf{k}_j(z)}(z)$. Note that this yields $R_j'(R_j(x)) = R_1'(R_1(x))$ and $\mathbf{k}_j(z) \in \mathbf{C}$.

Repeat the above construction for each P_i. Then define

$$\overline{T}_j = \begin{cases} R_j' & \text{on } A_{\mathbf{g}_j} \\ I_j & \text{elsewhere on } b_j(Q) \end{cases}.$$

Then $\overline{T}_j$ is a $\mathbf{C}$-partial iterate and $\sigma_{\mathbf{T}}(x, \overline{T}_1\overline{T}_2 \cdots \overline{T}_d(x))$ equals, for instance, $\sigma_{\mathbf{T}}(x, R_1'(R_1(x)) \in U$. □

The next lemma says that a given $\mathbf{C}$-partial speedup defined on a certain type of subset of Q, it can be extended to all of Q. These certain subsets of Q are defined as follows:

Definition 3.6. *Let $\mathbf{N} \geq \mathbf{0}$. We say a subset $\mathcal{B} \subseteq [\mathbf{N}]$ is* **lower triangular** *if for all $j = 1, ..., d$, $(\mathcal{B} - \mathbf{e}_j) \bigcap [\mathbf{N}] \subseteq \mathcal{B}$.*

Lemma 3.7. *Fix a cone $\mathbf{C}$ and suppose $\mathbf{T}^{\sigma}$ is an ergodic $G-$extension of the $\mathbb{Z}^d-$action $(X, \mathcal{X}, \mu, \mathbf{T})$. Suppose $\{A_{\mathbf{y}}\}_{\mathbf{y} \in Q}$ is a rectangular collection of size $(2, 2, ..., 2)$ and F is a lower triangular subset of Q. Suppose further that*

1. *for every $\mathbf{y} \in Q - F$, we are given an open subset $U_{\mathbf{y}} \subseteq G$;*
2. *for every set $A_{\mathbf{v}}$ with $\mathbf{v} \in b_j(F)$, there is a $\mathbf{C}-$partial iterate I_j of $\mathbf{T}$ taking $A_{\mathbf{v}}$ to $A_{\mathbf{v}+\mathbf{e}_j}$; and*
3. *the partial iterates defined in (2) commute, i.e. when $\mathbf{v} \in b_j(Q) \bigcap b_k(Q)$ has $\mathbf{v} + \mathbf{e}_j + \mathbf{e}_k \in F$, then $I_j \circ I_k = I_k \circ I_j$ a.s. on $A_{\mathbf{v}}$.*

Then there exists a $\mathbf{C}-$partial speedup $\overline{\mathbf{T}} = (\overline{T}_1, ..., \overline{T}_d)$ of $\mathbf{T}$ such that

- i) *$\overline{T}_j = I_j$ wherever the iterate I_j is defined;*
- ii) *$\{A_{\mathbf{y}}\}_{\mathbf{y} \in Q}$ is a speedup block for $\overline{\mathbf{T}}$; and*
- iii) *for every $\mathbf{y} \in Q - F$, $\sigma_{\mathbf{T}}(x, \overline{\mathbf{T}}_{\mathbf{y}}(x)) \in U_{\mathbf{y}}$ for almost every $x \in A_{\mathbf{0}}$.*

Proof. The proof is done inductively on the dimension of "subcubes" of Q containing $\mathbf{0}$. We begin by setting $\overline{T}_j = I_j$ where the $\mathbf{C}$-partial iterate I_j is defined.

Recall that $Q_1 = \{\mathbf{v} \in Q : v_1 + ... + v_d = 1\} = \{\mathbf{e}_1, ..., \mathbf{e}_d\}$. If there exists $\mathbf{e}_j \in Q_1 - F$, use Lemma 3.3 (with sets $A_{\mathbf{0}}$ and $A_{\mathbf{e}_j}$, $\mathbf{v} = \mathbf{0}$, and $U = U_{\mathbf{e}_j}$) to yield a $\mathbf{C}$-partial iterate $\overline{T}_j : A_{\mathbf{0}} \to A_{\mathbf{e}_j}$ with $\sigma_{\mathbf{T}}(x, \overline{T}_j(x)) \in U_{\mathbf{e}_j}$.

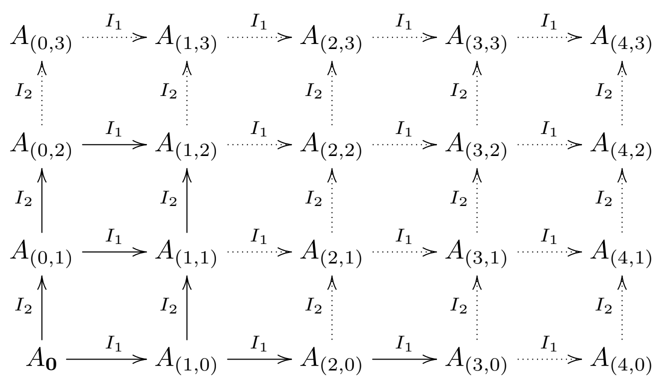
Next, "complete" all the faces of the cube containing the origin (a face can be thought of as a two-dimensional "subcube" of Q). More specifically, if there exists $\mathbf{y} \in Q_2 - F$, so $\mathbf{y} = \mathbf{e}_i + \mathbf{e}_j$, use Lemma 3.5 (with the Q in that statement equal to the two-dimensional cube $\{\mathbf{0}, \mathbf{e}_i, \mathbf{e}_j, \mathbf{y}\}$ and $U = U_{\mathbf{y}}$) to construct $\mathbf{C}$-partial iterates

$\overline{T}_i : A_{\mathbf{e}_j} \to A_{\mathbf{y}}$ and $\overline{T}_j : A_{\mathbf{e}_i} \to A_{\mathbf{y}}$ satisfying $\overline{T}_i \circ \overline{T}_j = \overline{T}_j \circ \overline{T}_i$ on $A_{\mathbf{0}}$ and $\sigma_{\mathbf{T}}(x, \overline{\mathbf{T}}_{\mathbf{y}}(x)) \in U_{\mathbf{y}}$.

Repeat this process for the three-dimensional subcubes containing the origin, then the four-dimensional subcubes, etc. More specifically, given that we have "completed" the $(k-1)$-dimensional subcubes containing the origin, we complete the k-dimensional subcubes containing the origin as follows: if there exists $\mathbf{y} \in Q_k - F$ of the form $\mathbf{y} = \mathbf{e}_{i_1} + ... + \mathbf{e}_{i_k}$, where the $\mathbf{e}_{i_j}$ are distinct elements from $\{\mathbf{e}_1, ...\mathbf{e}_d\}$, use Lemma 3.5 (with the Q in that statement being the k-dimensional cube in the dimensions i_1 through i_k and $U = U_{\mathbf{y}}$) to construct the remaining $\mathbf{C}$-partial iterates $\overline{T}_{i_j}$ on this cube which commute and for which $\sigma_{\mathbf{T}}(x, \overline{\mathbf{T}}_{\mathbf{y}}(x)) \in U_{\mathbf{y}}$.

After completing the d−dimensional subcube, we obtain a $\mathbf{C}$−partial speedup $\overline{\mathbf{T}}$ satisfying the conclusions of the lemma. □

The next lemma (Lemma 3.9) is the key result of this subsection. It essentially says that given a picture like the one below, where the partial iterates indicated by the solid arrows are already defined, we can construct the partial iterates indicated by the dashed arrows so that the diagram commutes and the cocycles associated to these iterates take values in prescribed open subsets of G.



The sets connected by the solid arrows above constitute an example of a "lower triangular speedup block". More generally:

Definition 3.8. *Let $(X, \mathcal{X}, \mu, \mathbf{T})$ be an ergodic $\mathbb{Z}^d$−action. Let $\mathcal{B}$ be a lower triangular subset of $[\mathbf{N}]$. We say $A_{\mathcal{B}} = \{A_{\mathbf{v}}\}_{\mathbf{v}\in\mathcal{B}}$ is a* **lower triangular speedup block (ltsb)** *if the sets $A_{\mathbf{v}}$ are disjoint, measurable, of the same positive measure, and*

1. *for every $j = 1, ..., d$, given any $\mathbf{v} \in b_j(\mathcal{B})$ there exists a partial iterate I_j of $\mathbf{T}$ taking $A_{\mathbf{v}}$ to $A_{\mathbf{v}+\mathbf{e}_j}$; and*
2. *for any $j, k \in \{1, ..., d\}$ with $j \neq k$, given any $\mathbf{v} \in b_j(\mathcal{B}) \bigcap b_k(\mathcal{B})$, $I_j \circ I_k(x) = I_k \circ I_j(x)$ for every $x \in A_{\mathbf{v}}$.*

We call the maps $I_1, I_2, ..., I_d$ the **iterates** *of the block $A_{\mathcal{B}}$.*

Given a cone $\mathbf{C} \subset \mathbb{Z}^d$, we say the ltsb is a **C-ltsb** *if the iterate functions of $I_1, ..., I_d$ take values only in $\mathbf{C}$.*

Lemma 3.9 (Completing Lemma)**.** *Fix a cone $\mathbf{C} \subset \mathbb{Z}^d$ and $\mathbf{N} \geq \mathbf{0}$, and suppose $\mathbf{T}^\sigma$ is an ergodic G−extension of the $\mathbb{Z}^d$−action $(X, \mathcal{X}, \mu, \mathbf{T})$. Let $\{A_{\mathbf{v}}\}_{\mathbf{v}\in[\mathbf{N}]}$ be a rectangular collection and suppose $\mathcal{B} \subseteq [\mathbf{N}]$ is such that $A_{\mathcal{B}} = \{A_{\mathbf{v}}\}_{\mathbf{v}\in\mathcal{B}}$ is a*

$\mathbf{C}-$lower triangular speedup block with iterates $I_1, ..., I_d$. Suppose that for every $\mathbf{v} \in [\mathbf{N}] - \mathcal{B}$, we are given an open subset $U_{\mathbf{v}} \subseteq G$. If $\mathbf{0} \in [\mathbf{N}] - \mathcal{B}$, we assume $e_G \in U_{\mathbf{0}}$.

Then there is a $\mathbf{C}-$partial speedup $\overline{\mathbf{T}} = (\overline{T}_1, ..., \overline{T}_d)$ of $\mathbf{T}$ such that

1. *$\overline{T}_j = I_j$ wherever the iterate I_j is defined;*
2. *$\{A_{\mathbf{v}}\}_{\mathbf{v}\in[\mathbf{N}]}$ is a (rectangular) speedup block for $\overline{\mathbf{T}}$; and*
3. *for every $\mathbf{v} \in [\mathbf{N}] - \mathcal{B}$, $\sigma_{\mathbf{T}}(x, \overline{\mathbf{T}}_{\mathbf{v}}(x)) \in U_{\mathbf{v}}$ for a.e. $x \in A_{\mathbf{0}}$.*

Proof. If $\mathcal{B} = [\mathbf{N}]$ then the lower triangular speedup block is already the desired (rectangular) speedup block and we are done. If $\mathcal{B} = \emptyset$, use Lemma 3.3 to construct a $\mathbf{C}$-partial iterate $I_1 : A_{\mathbf{0}} \to A_{\mathbf{e}_1}$ satisfying $\sigma_{\mathbf{T}}(x, I_1(x)) \in U_{\mathbf{e}_1}$ for almost every $x \in A_{\mathbf{0}}$. Rename $\mathcal{B}$ to be $\{\mathbf{0}, \mathbf{e}_1\}$ and continue as below.

For $r \in \{1, ..., d\}$, set

$$Q^{(r)} = \{(x_1, x_2, ..., x_r, 0, 0, ..., 0) \in \mathbb{Z}^d : x_j \in \{0, 1\} \text{ for } j = 1, ..., r\}.$$

Thus $Q^{(r)}$ is an r-dimensional cube sitting in $\mathbb{Z}^d$. Given a $d-$dimensional rectangular collection $\{A_{\mathbf{v}}\}_{\mathbf{v}\in[\mathbf{N}]}$, $\mathbf{w} \in [\mathbf{N}]$, and $1 \le r \le d$, set $Q^{(r)}_{\mathbf{w}}$ to be the $r-$dimensional rectangular collection $\{A_{\mathbf{v}+\mathbf{w}}\}_{\mathbf{v}\in Q^{(r)}}$ of size $(2, 2, ..., 2)$. We also set $\mathbf{1}^{(r)}$ equal to the vector in $Q^{(r)}$ with $x_j = 1$ for all $1 \le j \le r$.

We will prove this theorem by repeated application of Lemma 3.7, applied to larger and larger dimensional cubes. For the example shown in the picture above, we would first extend the $\mathbf{C}$-partial iterates to the 1-dimensional cube that is $\{(3, 0), (4, 0)\} = Q^{(1)}_{\mathbf{w}}$ with $\mathbf{w} = (3, 0)$. We would next extend the $\mathbf{C}$-partial iterates to the 2-dimensional cube that is $\{(1, 0), (1, 1), (2, 0), (2, 1)\} = Q^{(2)}_{\mathbf{w}}$ with $\mathbf{w} = (1, 0)$ and continue with the rest of the 2-dimensional cubes that make up the first row of the array. We would then extend to the second row, starting with the 2-dimensional cube $Q^{(2)}_{\mathbf{w}}$ with $\mathbf{w} = (1, 1)$ and moving, cube by cube, to the right until that row is complete. Next, we extend the $\mathbf{C}$-partial iterates to the 1-dimensional cube $\{(0, 2), (0, 3)\} = Q^{(1)}_{\mathbf{w}}$ with $\mathbf{w} = (0, 2)$. Finally, we would extend to $Q^{(2)}_{\mathbf{w}}$ with $\mathbf{w} = (0, 2)$ and continue along that row until the $\mathbf{C}$-partial iterates are defined on the entire rectangular collection, yielding the result.

To write out the general case, let r be the smallest natural number such that $(N_1-1, N_2-1, .., N_r-1, 0, 0, ..., 0) \notin \mathcal{B}$. Let $g_r = \max\{y : (N_1-1, N_2-1, ..., N_{r-1}-1, y, 0, ..., 0) \in \mathcal{B}\}$ and define $g_{r-1}, g_{r-2}, ..., g_1$ recursively by setting

$$g_j = \max\{y : (N_1 - 1, N_2 - 1, ..., N_{j-1} - 1, y, g_{j+1} + 1, ..., g_r + 1, 0, ..., 0) \in \mathcal{B}\}$$

If no such y exists, set $g_j = 0$. For the example shown above, $r = 1$ and $g_1 = 3$.

Set $\mathbf{g} = (g_1, g_2, ..., g_r, 0, ..., 0) \in [\mathbf{N}]$. Note that $\mathbf{g} \in \mathcal{B}$ and $\mathbf{g} + \mathbf{1}^{(r)} \notin \mathcal{B}$.

Now use Lemma 3.7 to extend the $\mathbf{C}$-partial iterates $I_1, ..., I_r$ to the rectangular collection $\{A_{\mathbf{v}}\}_{\mathbf{v}\in Q^{(r)}_{\mathbf{g}}}$. Let $\mathcal{B}' = \mathcal{B} \cup Q^{(r)}_{\mathbf{g}}$ and note that $\mathcal{B}'$ is lower triangular and strictly contains $\mathcal{B}$. The $\mathbf{C}$-partial iterates are now defined on more of the rectangular collection $\{A_{\mathbf{v}}\}_{\mathbf{v}\in[\mathbf{N}]}$ than before, and the portion they are defined on satisfies the hypotheses of this lemma.

Rename $\mathcal{B}'$ as $\mathcal{B}$ and repeat the above steps. As the size of $\mathcal{B}$ has increased and yet our rectangle $[\mathbf{N}]$ has finite size, this process will eventually end: this will occur when $r = d$ and $\mathcal{B} = [\mathbf{N}]$, yielding the result. □

Corollary 3.10. *Fix a cone $\mathbf{C} \subset \mathbb{Z}^d$ and suppose $\mathbf{T}^\sigma$ is an ergodic $G-$extension of the $\mathbb{Z}^d-$action $(X, \mathcal{X}, \mu, \mathbf{T})$. Let $\{A_\mathbf{v}\}_{\mathbf{v}\in[\mathbf{m}]}$ be any rectangular collection. Then, for any collection $\{U_\mathbf{v}\}_{\mathbf{v}\in[\mathbf{m}]}$ of open subsets of G with $e_G \in U_\mathbf{0}$, there is a $\mathbf{C}-$partial speedup $\overline{\mathbf{T}}$ of the original action $\mathbf{T}$ such that $\{A_\mathbf{v}\}_{\mathbf{v}\in[\mathbf{m}]}$ is a speedup block for $\overline{\mathbf{T}}$ and $\sigma_\mathbf{T}(x, \overline{\mathbf{T}}_\mathbf{v}(x)) \in U_\mathbf{v}$ for every $\mathbf{v} \in [\mathbf{m}]$, for $\mu-$a.e. $x \in A_\mathbf{0}$.*

Proof. Apply the previous result to the block $\mathcal{B} = \emptyset$. □

3.2. **L-collections.**

Definition 3.11. *Let $\mathbf{k}, \mathbf{h} \in (\mathbb{Z}_+)^d$. An **$L-$set** is a subset $L(\mathbf{k}, \mathbf{h})$ of $\mathbb{Z}^d$ of the form*

$$L(\mathbf{k}, \mathbf{h}) = [\mathbf{k}, \mathbf{k} + \mathbf{h} + \mathbf{1}] - [\mathbf{k} + \mathbf{1} + \mathbf{1}, \mathbf{k} + \mathbf{h} + \mathbf{1}].$$

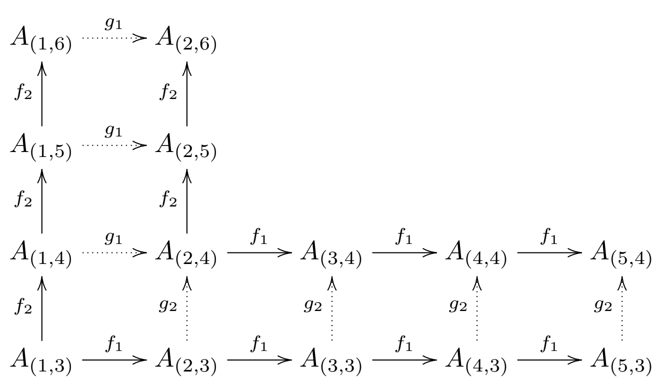
*Given an $L-$set $L(\mathbf{k}, \mathbf{h})$ and $j \in \{1, ..., d\}$, the $\mathbf{j}^{th}$ **side** of $L(\mathbf{k}, \mathbf{h})$ is the set of vectors $\mathbf{v} \in L(\mathbf{k}, \mathbf{h})$ satisfying $v_j = k_j$. The **outside** of $L(\mathbf{k}, \mathbf{h})$ is the set $Out(L)$ of vectors $\mathbf{v} \in L(\mathbf{k}, \mathbf{h})$ satisfying $v_j = k_j$ for some j; the **inside** of $L(\mathbf{k}, \mathbf{h})$ is the set $In(L)$ of vectors in $L(\mathbf{k}, \mathbf{h})$ not on the outside.*

Note that the vector $\mathbf{k}$ lies on all d sides of $L(\mathbf{k}, \mathbf{h})$.

Definition 3.12. *Let $(X, \mathcal{X}, \mu, \mathbf{T})$ be an ergodic $\mathbb{Z}^d-$action. An **$L-$collection** is a collection of pairwise disjoint measurable subsets $\{A_\mathbf{v}\}_{\mathbf{v}\in L(\mathbf{k},\mathbf{h})}$ of the same positive measure, where the indexing set $L(\mathbf{k}, \mathbf{h})$ is an $L-$set, together with partial iterates $f_1, f_2, ..., f_d$ of $\mathbf{T}$ satisfying $f_j(A_\mathbf{v}) = A_{\mathbf{v}+\mathbf{e}_j}$ if $\mathbf{v}$ and $\mathbf{v} + \mathbf{e}_j$ are both in the outside of $L(\mathbf{k}, \mathbf{h})$, or both in the inside of $L(\mathbf{k}, \mathbf{h})$.*

We define $\mathbf{C} - L-$collections in the obvious way: they are $L-$collections where the partial iterates $f_1, f_2, ..., f_d$ of $\mathbf{T}$ have iterate functions which take values only in cone $\mathbf{C}$.

As an example, the $2-$dimensional $L-$collection with $\mathbf{k} = (1, 3)$ and $\mathbf{h} = (4, 3)$ is the following collection of sets and partial iterates indicated below by the solid arrows (this picture explains why we use the terminology "$L-$collection"):



The next lemma says that given an $L-$collection, we can "quilt" the outside to the inside via partial iterates. For instance, in the previous example, we can construct the iterates indicated by the dashed arrows in the above picture.

Lemma 3.13. *Fix a cone $\mathbf{C} \subset \mathbb{Z}^d$, an open set $U \subseteq G$, and $\mathbf{N} \geq \mathbf{0}$. Let $\mathbf{T}^\sigma$ be an ergodic $G-$extension of the $\mathbb{Z}^d-$action $(X, \mathcal{X}, \mu, \mathbf{T})$. Suppose that:*

1. *$\{A_\mathbf{v}\}_{\mathbf{v}\in[\mathbf{N}]}$ is a rectangular collection;*
2. *$L(\mathbf{k}, \mathbf{h})$ is an L-set with $L(\mathbf{k}, \mathbf{h}) \subset [\mathbf{N}]$;*
3. *$\{A_\mathbf{v}\}_{\mathbf{v}\in L(\mathbf{k},\mathbf{h})}$ is a $\mathbf{C} - L-$collection with partial iterates $f_1, ..., f_d$; and*
4. *for every $\mathbf{v} \in Out(L)$, there is a partial iterate $I_\mathbf{v}$ taking $A_\mathbf{0}$ to $A_\mathbf{v}$, and these iterates satisfy $f_s \circ I_\mathbf{v} = I_{\mathbf{v}+\mathbf{e}_s}$ a.s. on $A_\mathbf{0}$ whenever $\{\mathbf{v}, \mathbf{v} + \mathbf{e}_s\} \subseteq Out(L)$.*

Then there are $\mathbf{C}-$partial iterates $g_1, ..., g_d$ of $\mathbf{T}$ such that:

- i) *for each $\mathbf{v}$ which is in the j^{th} side of $L(\mathbf{k}, \mathbf{h})$ but not in any other side, g_j takes $A_\mathbf{v}$ to $A_{\mathbf{v}+\mathbf{e}_j}$;*
- ii) *the maps f_j and g_k commute, i.e. if $\mathbf{v} \in Out(L)$ and $\mathbf{v} + \mathbf{e}_j \in Out(L)$ but $\mathbf{v} + \mathbf{e}_k \in In(L)$, then $f_j \circ g_k = g_k \circ f_j$ a.s. on $A_\mathbf{v}$;*
- iii) *for a.e. $x \in A_\mathbf{k}$, the base of the L-collection, and for any permutation ρ of $\{1, ..., d\}$, we have $g_d \circ f_{d-1} \circ \cdots \circ f_1(x) = g_{\rho(d)} \circ f_{\rho(d-1)} \circ \cdots \circ f_{\rho(1)}(x)$; and*
- iv) *for a.e. $x \in A_\mathbf{0}$, $\sigma_\mathbf{T}(x, g_1 \circ I_{\mathbf{k}+\mathbf{1}-\mathbf{e}_1}(x)) \in U$.*

Proof. The proof is divided into three parts. First, we partition the sets $A_\mathbf{k}$ and $A_{\mathbf{k}+\mathbf{1}}$ so that various iterate functions are constant on the atoms. We then work inductively on these atoms, first to find an appropriate set $\mathbf{C}_\mathbf{w} \subset \mathbb{Z}^d$ and then to use this set as we apply Lemma 3.4 to find one of our partial iterates, namely g_1. This will be one "stitch" between the outside and the inside of the $\mathbf{C} - L$-collection, and the final step of our proof is to use this stitch to define the other partial iterates that will "quilt" the $\mathbf{C} - L-$collection together. Now for the details:

For any $\mathbf{v} = (v_1, ..., v_d) \in L(\mathbf{k}, \mathbf{h})$, some (possibly none or all) of the partial iterates $f_1, ..., f_d$ are defined on $A_\mathbf{v}$. Let $\mathcal{P}_\mathbf{v}$ be a finite or countable partition of $A_\mathbf{v}$ so that on each atom of $\mathcal{P}_\mathbf{v}$, all the iterate functions $\mathbf{f}_1, \mathbf{f}_2, ..., \mathbf{f}_d : A_\mathbf{v} \to \mathbf{C}$ of the partial iterates defined on $A_\mathbf{v}$ are constant. Then for $\mathbf{v} \in Out(L)$, $f_1^{-(v_1-k_1)} \circ f_2^{-(v_2-k_2)} \circ \cdots \circ f_d^{-(v_d-k_d)}\mathcal{P}_\mathbf{v}$ is the pullback of this partition onto $A_\mathbf{k}$, the base of the L-collection. Set

$$\mathcal{P}_{out} = \bigvee_{\mathbf{v}\in Out(L)} f_1^{-(v_1-k_1)} \circ \cdots \circ f_d^{-(v_d-k_d)}(\mathcal{P}_\mathbf{v}).$$

Similarly, for $\mathbf{v} \in In(L)$, $f_1^{-(v_1-(k_1+1))} \circ \cdots \circ f_d^{-(v_d-(k_d+1))}\mathcal{P}_\mathbf{v}$ is the pullback of this partition onto $A_{\mathbf{k}+\mathbf{1}}$. Set

$$\mathcal{P}_{in} = \bigvee_{\mathbf{v}\in In(L)} f_1^{-(v_1-(k_1+1))} \circ \cdots \circ f_d^{-(v_d-(k_d+1))}(\mathcal{P}_\mathbf{v}).$$

Then $\mathcal{P}_{out}$ and $\mathcal{P}_{in}$ are partitions on $A_\mathbf{k}$ and $A_{\mathbf{k}+\mathbf{1}}$, respectively. Denote the atoms of $\mathcal{P}_{out}$ by $B_1, B_2, ...$ and arbitrarily partition $A_{\mathbf{k}+\mathbf{1}}$ into sets $B'_1, B'_2, ...$ such that $\mu(B'_j) = \mu(B_j)$ for all j.

Consider the partition $\mathcal{P}_{in}|_{B'_1}$: list the positive-measure elements of this partition of $B'_1 \subset A_{\mathbf{k}+\mathbf{1}}$ as $C'_1, C'_2, \ldots$. Partition B_1 arbitrarily into sets $C_1, C_2, ...$ with $\mu(C_j) = \mu(C'_j)$ for all j. Note that for each j, we have

- for any $\mathbf{v} \in Out(L)$, the iterate function of $f_1^{v_1-k_1} \circ \cdots \circ f_d^{v_d-k_d}$ is constant on C_j: set $\mathbf{a}_{j,\mathbf{v}}$ to be this constant;
- for any $\mathbf{v} \in In(L)$, the iterate function of $f_1^{v_1-(k_1+1)} \circ \cdots \circ f_d^{v_d-(k_d+1)}$ is constant on C'_j: set $\mathbf{b}_{j,\mathbf{v}}$ to be this constant.

Now fix j. Let $\mathbf{w} \in \mathbb{Z}^d$ be a vector such that

$$\mathbf{C_w} \subset \mathbf{C} \cap (\mathbf{C} + \mathbf{a}_{j,\mathbf{v}} - \mathbf{a}_{j,(\mathbf{k+1-e}_1)} - \mathbf{b}_{j,(\mathbf{v}+\mathbf{e}_s)})$$

for every $\mathbf{v}$ and s such that $\mathbf{v} \in Out(L)$ and $\mathbf{v} + \mathbf{e}_s \in In(L)$. Let $D_j = f_2 \circ f_3 \circ \cdots \circ f_d(C_j) \subseteq A_{\mathbf{k+1-e}_1}$. Now use Lemma 3.4 with sets C_j, D_j, and C_j', partial iterate $f_2 \circ \cdots \circ f_d$, and vector $\mathbf{w}$ from above: this defines the partial iterate $g_1 : D_j \to C_j'$ whose iterate function $\mathbf{g_1}$ takes values only in $\mathbf{C_w}$ (thus in $\mathbf{C}$) and for which $\sigma_{\mathbf{T}}(x, g_1 \circ I_{\mathbf{k+1-e}_1}(x)) \in U$ for almost every $x \in A_{\mathbf{0}}$.

What remains is for us to define the other g_i's and the rest of g_1. Let $\mathbf{v}$ and s be such that $\mathbf{v} \in Out(L)$ and $\mathbf{v} + \mathbf{e}_s \in In(L)$. We need to define g_s on $f_1^{v_1-k_1} \circ \cdots \circ f_d^{v_d-k_d}(C_j)$, which we do by moving a point in this domain first back to $C_j \subset A_{\mathbf{k}}$, then moving it to the set D_j, where we can use the already-defined g_1 to move it to C_j', and finally moving it to $A_{\mathbf{v}+\mathbf{e}_s}$. In other words, for $z \in f_1^{v_1-k_1} \circ \cdots \circ f_d^{v_d-k_d}(C_j)$, set

$$g_s(z) = (f_1^{u_1} \circ \cdots \circ f_d^{u_d}) \circ g_1 \circ (f_2 \circ \cdots \circ f_d) \circ (f_1^{-(v_1-k_1)} \circ \cdots \circ f_d^{-(v_d-k_d)})(z),$$

where $\mathbf{u} = \mathbf{v} + \mathbf{e}_s - (\mathbf{k} + \mathbf{1})$.

The iterate function associated to g_s is

$$\mathbf{g}_s = \mathbf{b}_{j,(\mathbf{v}+\mathbf{e}_s)} + \mathbf{g_1} + \mathbf{a}_{j,(\mathbf{k+1-e}_1)} - \mathbf{a}_{j,\mathbf{v}}.$$

Thus $\mathbf{g}_s \in \mathbf{C}$ exactly when $\mathbf{g}_1 \in \mathbf{C} + \mathbf{a}_{j,\mathbf{v}} - \mathbf{a}_{j,(\mathbf{k+1-e}_1)} - \mathbf{b}_{j,(\mathbf{v}+\mathbf{e}_s)}$, which follows since $\mathbf{g}_1 \in \mathbf{C_w}$.

Having now defined the g_s on the images of C_j, repeat the argument for each j to define the iterates on all the appropriate images of B_1. Then repeat this argument for $B_2, B_3, ...$; this produces partial iterates $\{g_s\}_{s=1}^d$ which satisfy the conclusions of the lemma. □

3.3. **Completing the proof of Theorem 3.1.** The next result tells us that if we are given a rectangular collection where commuting partial iterates have been defined on some lower triangular set, and if we are given a rectangular speedup block within the rectangular collection which is disjoint from the lower triangular set, then we can extend the partial iterates to a larger lower triangular block, encompassing both the original lower triangular set and the given speedup block. By repeating the argument in this lemma, we obtain a construction which establishes Theorem 3.1 in the case where the functions $g_{\mathbf{v}}$ are constant. Then by approximating the $g_{\mathbf{v}}$ by step functions we obtain a proof of Theorem 3.1.

Observe first that given a 2−dimensional lower triangular subset (or ltsb) of $[(N_1, N_2)]$, there exists a nonincreasing function

$$J_1 : \{0, 1, 2, ..., N_1 - 1\} \to \{-1, 0, 1, 2, ..., N_2 - 1\}$$

such that

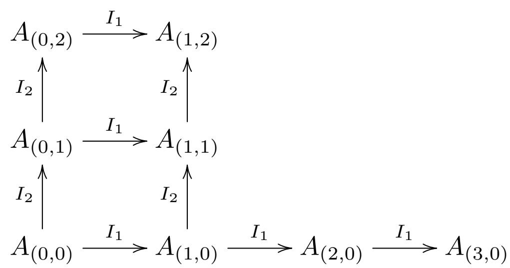
$$(x, y) \in \mathcal{B} \Leftrightarrow (x \geq 0 \text{ and } 0 \leq y \leq J_1(x)).$$

Note that if $J_1(x) = -1$, then $(x, y) \notin \mathcal{B}$ for any y.

Similarly, given a d−dimensional lower triangular subset (or ltsb) of $[\mathbf{N}] = [(N_1, ..., N_d)]$, there exists a sequence of functions $J_1, ..., J_{d-1}$ called the **height functions** of $\mathcal{B}$ such that

(1) for each $r \in \{1, ..., d-1\}$, J_r maps $[(N_1, ..., N_r)]$ into the finite set $\{-1, 0, ..., N_{r+1} - 1\}$;
(2) each J_r is non-increasing along any one coordinate if the other coordinates are kept fixed; and
(3) $\mathbf{v} \in \mathcal{B} \Leftrightarrow (\mathbf{v} \geq 0$ and for each $r \in \{2, ..., d\}, v_r \leq J_{r-1}(v_1, ..., v_{r-1}))$.

As an example, if $J_1(0) = J_1(1) = 2$, $J_1(2) = J_1(3) = 0$ and $J_1(4) = -1$, the (2−dimensional) ltsb with height function J_1 is the collection of sets shown below with iterates I_1 and I_2 (here $\mathbf{N}$ can be any integer vector greater than or equal to $(5, 3)$):



Lemma 3.14 (Iterative Filling Lemma)**.** *Fix a cone* $\mathbf{C} \subset \mathbb{Z}^d$*. Fix* $\mathbf{N}$*,* $\mathbf{k}$*,* $\mathbf{h} \in \mathbb{Z}^d$ *where* $\mathbf{N} > \mathbf{k} + \mathbf{h}$*,* $\mathbf{h} > \mathbf{0}$*, and* $\mathbf{k} \geq \mathbf{0}$*. Let* $\mathbf{T}^\sigma$ *be an ergodic* $G-$*extension of the* $\mathbb{Z}^d-$*action* $(X, \mathcal{X}, \mu, \mathbf{T})$ *and* $\{A_\mathbf{v}\}_{\mathbf{v}\in[\mathbf{N}]}$ *be a rectangular collection in* X*.*
Suppose that $\mathcal{B} \subseteq [\mathbf{N}]$ *is such that:*

- $A_\mathcal{B} = \{A_\mathbf{v}\}_{\mathbf{v}\in\mathcal{B}}$ *is a* $\mathbf{C}-$*ltsb with partial iterates* $I_1, \ldots, I_d$*, and*
- *the last height function* J_{d-1} *of* $\mathcal{B}$ *satisfies* $J_{d-1}(k_1, ..., k_{d-1}) < k_d$*.*

Suppose also that

- *the sets* $\{A_\mathbf{v}\}_{\mathbf{v}\in[\mathbf{k},\mathbf{k}+\mathbf{h}]}$ *form a speedup block for a* $\mathbf{C}-$*partial speedup* $\overline{\mathbf{T}} = (\overline{T}_1, ..., \overline{T}_d)$ *of* $\mathbf{T}$*, and*
- *we are given, for every* $\mathbf{v} \in [\mathbf{N}] - \mathcal{B}$*, an open subset* $U_\mathbf{v} \subseteq G$*. If* $\mathbf{0} \in [\mathbf{N}] - \mathcal{B}$*, we assume* $e_G \in U_\mathbf{0}$*.*

Then, if we define for each $j \in \{1, ..., d-1\}$*,* $\widetilde{J}_j : [(N_1, ..., N_j)] \to \{-1, 0, ..., N_{j+1} - 1\}$ *by*

$$\widetilde{J}_j(x_1, ..., x_j) = \begin{cases} \max\{J_j(x_1, ..., x_j), k_{j+1} + h_{j+1} - 1\} & \text{if } (x_1, ..., x_j) < (k_1 + h_1, ..., k_j + h_j) \\ J_j(x_1, ..., x_j) & \text{otherwise} \end{cases}$$

and let $\widetilde{\mathcal{B}}$ *be the lower triangular subset with height functions* $\widetilde{J}_1, ..., \widetilde{J}_{d-1}$*, the sets* $\{A_\mathbf{v}\}_{\mathbf{v}\in\widetilde{\mathcal{B}}}$ *form a* $\mathbf{C}-$*ltsb* $A_{\widetilde{\mathcal{B}}}$ *with partial iterates* $\widetilde{I}_1, \ldots, \widetilde{I}_d$ *which simultaneously extend the partial iterates associated to the block* $A_\mathcal{B}$ *and the partial speedup* $\overline{\mathbf{T}}$*, i.e.*

i) *for every* $\mathbf{v} \in b_j(\mathcal{B}), \widetilde{I}_j = I_j$ *on* $A_\mathbf{v}$*, and*
ii) *for every* $\mathbf{v} \in [\mathbf{k}, \mathbf{k} + \mathbf{h} - \mathbf{e}_j]$*,* $\widetilde{I}_j = \overline{T}_j$ *on* $A_\mathbf{v}$*.*

Furthermore, given any $\mathbf{v} = (v_1, ..., v_d) \in \left(\widetilde{\mathcal{B}} - \mathcal{B} - [\mathbf{k}, \mathbf{k} + \mathbf{h}]\right) \bigcup \{\mathbf{k}\}$*, we have*

$$\sigma_\mathbf{T}(x, \widetilde{I}_1^{v_1} \circ \cdots \circ \widetilde{I}_d^{v_d}(x)) \in U_\mathbf{v}$$

for a.e. $x \in A_\mathbf{0}$*.*

Proof. First, use Lemma 3.9 to find a partial speedup $\mathbf{R}$ extending the iterates $I_1, ..., I_d$ of $A_{\mathcal{B}}$ such that $\{A_{\mathbf{v}}\}_{\mathbf{v}\in[\mathbf{N}]}$ is a speedup block for $\mathbf{R}$, where for each $\mathbf{v} \in [\mathbf{N}] - \mathcal{B}$, $\sigma_{\mathbf{T}}(x, \mathbf{R}_{\mathbf{v}}(x)) \in U_{\mathbf{v}}$ for a.e. $x \in A_{\mathbf{0}}$.

Consider the $L-$set $L(\mathbf{k}-\mathbf{1}, \mathbf{h})$. Restricting the action $\mathbf{R}$ to the sets $A_{\mathbf{v}}$ where $\mathbf{v} \in Out(L)$, and restricting the action $\overline{\mathbf{T}}$ to the sets $A_{\mathbf{v}}$ where $\mathbf{v} \in In(L)$ turns $\{A_{\mathbf{v}}\}_{\mathbf{v}\in L}$ into an $L-$collection. By Lemma 3.13 there are partial iterates $g_1, ..., g_d$ mapping sets associated to vectors in $Out(L)$ to sets associated to respective vectors in $In(L)$, with $\sigma_{\mathbf{T}}(x, g_1 \circ \mathbf{R}_{\mathbf{k}+\mathbf{1}-\mathbf{e}_1}(x)) \in U_{\mathbf{k}}$ for a.e. $x \in A_{\mathbf{0}}$.

Now, if we define, for each $j \in \{1, ..., d\}$, maps $\widetilde{I}_1, ..., \widetilde{I}_d$ so that:

- $\widetilde{I}_j$ coincides with g_j on $A_{\mathbf{v}}$ whenever $\mathbf{v} \in Out(L)$ and $\mathbf{v} + \mathbf{e}_j \in In(L)$;
- $\widetilde{I}_j$ coincides with $\overline{T}_j$ on $A_{\mathbf{v}}$ whenever $\mathbf{v} \in [\mathbf{k}, \mathbf{k}+\mathbf{h}-\mathbf{e}_j]$; and
- $\widetilde{I}_j$ coincides with R_j on all other $\mathbf{v} \in b_j(\widetilde{\mathcal{B}})$;

then the iterates $\widetilde{I}_1, ..., \widetilde{I}_d$ satisfy the conclusions of the lemma. □

Lemma 3.15. *Fix* $\mathbf{N} \geq \mathbf{0}$, $\mathbf{h} \geq \mathbf{0}$, *and a cone* $\mathbf{C} \subset \mathbb{Z}^d$. *Let* $\mathbf{T}^\sigma$ *be an ergodic* $G-$*extension of the* $\mathbb{Z}^d-$*action* $(X, \mathcal{X}, \mu, \mathbf{T})$ *and let* $\{A_{\mathbf{v}}\}_{\mathbf{v}\in[\mathbf{N}]}$ *be a rectangular collection in* X.

Suppose that there are vectors $\mathbf{k}_1, \mathbf{k}_2, ..., \mathbf{k}_r \in [\mathbf{N}-\mathbf{h}]$ *such that the sets* $C_j = \bigcup_{\mathbf{v}\in[\mathbf{h}]} A_{\mathbf{k}_j+\mathbf{v}}$ *are pairwise disjoint, and that each* C_j *is a speedup block for a* $\mathbf{C}-$*partial speedup* $\overline{\mathbf{T}}_j$ *of* $\mathbf{T}$. *Suppose also that for all* $\mathbf{v} \in [\mathbf{N}]$, *we are given an open set* $U_{\mathbf{v}} \subseteq G$ *where* $e_G \in U_{\mathbf{0}}$.

Then there is a $\mathbf{C}-$*partial speedup* $\overline{\mathbf{T}}$ *extending the* $\overline{\mathbf{T}}_j$ *such that* $\{A_{\mathbf{v}}\}_{\mathbf{v}\in[\mathbf{N}]}$ *is a speedup block for* $\overline{\mathbf{T}}$ *and for all* $\mathbf{v}$ *in* $\left([\mathbf{N}] - \bigcup_{j=1}^r C_j\right) \cup \{\mathbf{k}_1, \ldots, \mathbf{k}_d\}$, *we have* $\sigma_{\mathbf{T}}(x, \overline{\mathbf{T}}_{\mathbf{v}}(x)) \in U_{\mathbf{v}}$ *for a.e.* $x \in A_{\mathbf{0}}$.

Proof. We will begin by using the last lemma with $\mathcal{B} = \emptyset$ and one of the speedup blocks, say C_1, to yield a lower triangular speedup block with iterates that extend $\overline{\mathbf{T}}_1$. We will then repeat this, using the last lemma again with this new lower triangular speedup block and another speedup block, say C_2.

So that we can continue this process, we need to order the C_j in such a way that when the lower triangular speedup block is increased to include the next C_j, the unincorporated C_j's are left entirely disjoint from the new, larger, lower triangular speedup block. This leads us to define

$$(v_1, \ldots, v_d) \prec (w_1, \ldots, w_d) \text{ iff } \begin{cases} \frac{|\mathbf{h}|}{h_1}v_1 + \cdots + \frac{|\mathbf{h}|}{h_d}v_d < \frac{|\mathbf{h}|}{h_1}w_1 + \cdots + \frac{|\mathbf{h}|}{h_d}w_d \\ \text{or} \\ \frac{|\mathbf{h}|}{h_1}v_1 + \cdots + \frac{|\mathbf{h}|}{h_d}v_d = \frac{|\mathbf{h}|}{h_1}w_1 + \cdots + \frac{|\mathbf{h}|}{h_d}w_d \text{ and} \\ \quad (v_1, \ldots, v_{d-1}) \prec (w_1, \ldots, w_{d-1}) \end{cases}$$

where $v_1 \prec w_1$ means $v_1 < w_1$. We renumber $\mathbf{k}_1, \mathbf{k}_2, ..., \mathbf{k}_r$ as necessary so that $\mathbf{k}_i \prec \mathbf{k}_{i+1}$ for $1 \leq i < r$.

Now we can begin as described above. Apply the Iterative Filling Lemma (Lemma 3.14) with $\mathcal{B}$ the empty set and $\mathbf{k} = \mathbf{k}_1$. We obtain a $\mathbf{C}$-ltsb $A_{\mathcal{B}_1}$ with iterates $\overline{\mathbf{T}}$ which extend $\overline{\mathbf{T}}_1$ on speedup block C_1 and such that for every $\mathbf{v} \in (\mathcal{B}_1 - [\mathbf{k}_1, \mathbf{k}_1 + \mathbf{h}]) \cup \{\mathbf{k}_1\}$, $\sigma_{\mathbf{T}}(x, \overline{\mathbf{T}}_{\mathbf{v}}x) \in U_{\mathbf{v}}$ for a.e. $x \in A_{\mathbf{0}}$.

Apply the Iterative Filling Lemma again, with $\mathcal{B} = \mathcal{B}_1$ and $\mathbf{k} = \mathbf{k}_2$ to obtain a $\mathbf{C}$-ltsb $A_{\mathcal{B}_2}$ whose iterates extend both $\overline{\mathbf{T}}$ on $\mathcal{B}_1$ and $\overline{\mathbf{T}}_2$ on speedup block C_2.

For each $\mathbf{v} \in (\mathcal{B}_2 - [\mathbf{k}_1, \mathbf{k}_1 + \mathbf{h}] - [\mathbf{k}_2, \mathbf{k}_2 + \mathbf{h}]) \cup \{\mathbf{k}_1, \mathbf{k}_2\}$, $\sigma_{\mathbf{T}}(x, \overline{\mathbf{T}}_{\mathbf{v}} x) \in U_{\mathbf{v}}$ for a.e. $x \in A_{\mathbf{0}}$.

We can continue in this fashion, applying the Iterative Filling Lemma repeatedly to obtain larger and larger lower triangular speedup blocks. Eventually we obtain a $\mathbf{C}$−ltsb $A_{\mathcal{B}_r}$ containing all the C_j, where the iterates of $A_{\mathcal{B}_r}$ coincide with the components of $\overline{\mathbf{T}}_j$ on each C_j and for every $\mathbf{v}$ in $\left(\mathcal{B}_r - \bigcup_{j=1}^r [\mathbf{k}_j, \mathbf{k}_j + \mathbf{h}]\right) \bigcup \{\mathbf{k}_1, ..., \mathbf{k}_r\}$, $\sigma_{\mathbf{T}}(x, \overline{\mathbf{T}}_{\mathbf{v}}(x)) \in U_{\mathbf{v}}$ for almost every $x \in A_{\mathbf{0}}$. Apply the Completing Lemma (Lemma 3.9) to $A_{\mathcal{B}_r}$ to complete the construction of $\overline{\mathbf{T}}$. □

We now complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Choose a neighborhood V of e_G such that $VV^{-1} \subseteq U$. Partition $A_{\mathbf{0}}$ into measurable sets $B_1, B_2, ...$ such that for each for every $x \in B_i$ and each $\mathbf{v}$ in

$$\left([\mathbf{N}] - \bigcup_{j=1}^{r} C_j\right) \cup \{\mathbf{k}_1, \ldots, \mathbf{k}_r\},$$

there is a group element $g_{i,\mathbf{v}}$ such that $g_{\mathbf{v}}(x) \in V\, g_{i,\mathbf{v}}$.

For each $\mathbf{v} \in ([\mathbf{N}] - \cup_{j=1}^r C_j) \cup \{\mathbf{k}_1, \ldots, \mathbf{k}_r\}$, partition $A_{\mathbf{v}}$ into disjoint sets $B_{\mathbf{v},i}$ such that $\mu(B_{\mathbf{v},i}) = \mu(B_i)$ for each i.

Next, define the rectangular collection $\{D_{\mathbf{v}}^{(i)}\}_{\mathbf{v}\in[\mathbf{N}]}$ by setting

- $D_{\mathbf{0}}^{(i)} = B_i$;
- $D_{\mathbf{v}}^{(i)} = B_{\mathbf{v},i}$ when $\mathbf{v} \in ([\mathbf{N}] - \cup_{j=1}^r C_j) \cup \{\mathbf{k}_1, \ldots, \mathbf{k}_r\}$; and
- $D_{\mathbf{v}}^{(i)} = (\overline{\mathbf{T}}_j)_{\mathbf{v}-\mathbf{k}_j}(B_{\mathbf{k}_j,i})$ if $\mathbf{v} \in [\mathbf{k}_j, \mathbf{k}_j + \mathbf{h}]$.

Apply Lemma 3.15 with the same cone $\mathbf{C}$, the rectangular collection $\{D_{\mathbf{v}}^{(i)}\}_{\mathbf{v}\in[\mathbf{N}]}$, speedup blocks $\{(\overline{\mathbf{T}}_j)_{\mathbf{v}}(B_{\mathbf{k}_j,i})\}_{\mathbf{v}\in[\mathbf{h}]}$, and $U_{\mathbf{v}} = V g_{i,\mathbf{v}}$ to produce a $\mathbf{C}$−partial speedup $\overline{\mathbf{T}}^{(i)}$ extending the $\overline{\mathbf{T}}_j$ such that $\{D_{\mathbf{v}}^{(i)}\}_{\mathbf{v}\in[\mathbf{N}]}$ is a speedup block for $\overline{\mathbf{T}}^{(i)}$ and given almost every $x \in B_i$, we have $\sigma_{\mathbf{T}}(x, (\overline{\mathbf{T}}^{(i)})_{\mathbf{v}}(x)) \in V g_{i,\mathbf{v}}$ for every $\mathbf{v} \in \left([\mathbf{N}] - \bigcup_j C_j\right) \bigcup \{\mathbf{k}_1, \ldots, \mathbf{k}_d\}$.

But for such a $\mathbf{v}$, we have $\sigma_{\mathbf{T}}(x, (\overline{\mathbf{T}}^{(i)})_{\mathbf{v}}(x))(g_{\mathbf{v}}(x))^{-1} \in (V g_{i,\mathbf{v}})(V g_{i,\mathbf{v}})^{-1} = VV^{-1} \subseteq U$ as desired. Setting $\overline{\mathbf{T}}$ so that it coincides with $\overline{\mathbf{T}}^{(i)}$ on the rectangular collection $\{D_{\mathbf{v}}^{(i)}\}_{\mathbf{v}\in[\mathbf{N}]}$ produces the speedup with the desired properties. □

4. Proof of Theorem 1.1

We now turn to proving our central result. The idea of the argument is this: we use Lemma 2.8 to find an increasing sequence of castles for $(Y, \mathcal{Y}, \nu, \mathbf{S})$; for each of these castles we construct a rectangular collection in X. Using Theorem 3.1 we realize these rectangular collections as speedup blocks for $\mathbf{C}$−partial speedups

$\mathbf{T}^1, \mathbf{T}^2, ...$ of $\mathbf{T}$, where each speedup extends the last and is defined on more of the space X. The corresponding $G-$extensions of these speedups will increase to a speedup $\overline{\mathbf{T}}^{\overline{\sigma}}$ of $\mathbf{T}^\sigma$ which satisfies the conclusions of Theorem 1.1.

Proof of Theorem 1.1. Recall that we have $\mathbf{T}^\sigma$ and $\mathbf{S}^\sigma$, $G-$extensions of the respective $\mathbb{Z}^d-$actions $(X, \mathcal{X}, \mu, \mathbf{T})$ and $(Y, \mathcal{Y}, \nu, \mathbf{S})$ with $\mathbf{T}^\sigma$ ergodic and $\mathbf{S}$ aperiodic. As G is locally compact, we can find a complete, right-invariant metric ρ compatible with the topology on G (there need not be a two-sided invariant metric compatible with the topology, see [B]). Choose $\epsilon > 0$ such that $B_\epsilon(e_G)$, the closed ball of $\rho-$radius ϵ centered at the identity, is compact and contained in U. Let ϵ_k be a decreasing sequence of positive real numbers satisfying $\sum_{k=1}^\infty \epsilon_k < \frac{\epsilon}{4}$.

Step 1: Preliminaries. For each k, choose a compact neighborhood U_k of the identity such that $U_k U_k^{-1} \subseteq B_{\epsilon_k}(e_G)$. Using Lemma 2.8, choose a sequence $\{\mathcal{C}_k^{\mathbf{S}}\}_{k=1}^\infty$ of castles for $\mathbf{S}$ with respect to these U_k. For each k, let $\{\tau_{k,j}^{\mathbf{S}}\}_j$ denote the towers comprising the castle $\mathcal{C}_k^{\mathbf{S}}$, let $\mathbf{N}_k$ be the common height of these towers, and let $A_{k,j,\mathbf{v}}^{\mathbf{S}}$ be the level at height $\mathbf{v}$ of tower $\tau_{k,j}^{\mathbf{S}}$. Observe that from Lemma 2.8 we obtain, for each k, j and $\mathbf{v}$, a group element $g_{k,j,\mathbf{v}} \in G$ such that for all $y \in A_{k,j,\mathbf{0}}^{\mathbf{S}}$,

$$\sigma_{\mathbf{S}}(y, \mathbf{v}) \in U_k \, g_{k,j,\mathbf{v}}$$

(in particular, $g_{k,j,\mathbf{0}}$ can be taken to be e_G for every k and j). Thus $U_k \, g_{k,j,\mathbf{v}}$ contains all values of the cocycle associated to movement from the base of $\tau_{k,j}^S$ to height $\mathbf{v}$ in the tower. Next, define for each k the set $\widetilde{K}_k = \bigcup_j \bigcup_{\mathbf{v} \in [\mathbf{N}_k]} U_k g_{k,j,\mathbf{v}}$ and let

$$K_k = B_\epsilon(e_G) \, \widetilde{K}_k \, (\widetilde{K}_k)^{-1} \, B_\epsilon(e_G);$$

observe that K_k is compact for all k. If $y \in |\mathcal{C}_k^{\mathbf{S}}| \bigcap \mathbf{S}_{-\mathbf{w}}(|\mathcal{C}_k^{\mathbf{S}}|)$, then $y \in A_{k,j,\mathbf{v}}^{\mathbf{S}}$ for some j and $\mathbf{v}$ and

$$\sigma_{\mathbf{S}}(y, \mathbf{w}) = \sigma_{\mathbf{S}}(\mathbf{S}_{-\mathbf{v}} y, \mathbf{v} + \mathbf{w}) \, \sigma_{\mathbf{S}}(\mathbf{S}_{-\mathbf{v}} y, \mathbf{v})^{-1} \in \widetilde{K}_k (\widetilde{K}_k)^{-1} \subseteq K_k.$$

Thus K_k contains all values of the cocycle $\sigma_{\mathbf{S}}$ associated to the k^{th} castle $\mathcal{C}_k^{\mathbf{S}}$. As we will later see, the use of the balls $B_\epsilon(e_G)$ will ensure K_k contains all values of the cocycle $\sigma_{\mathbf{T}}$ associated to the k^{th} partially defined speedup of $\mathbf{T}$.

By the uniform continuity of the inverse function and of group multiplication on the compact set $B_\epsilon(e_G) \times K_k \times B_\epsilon(e_G)$, we can choose for each k a constant $\delta_k > 0$ such that if $h_1, h_2, h_3, h_4 \in B_\epsilon(e_G)$ are such that $\rho(h_1, h_2) < \delta_k$, $\rho(h_3, h_4) < \delta_k$, and $g \in K_k$, then

$$\rho(h_1 g h_3, h_2 g h_4) < \frac{1}{k} \quad \text{and} \quad \rho(h_1 g h_3^{-1}, h_2 g h_4^{-1}) < \frac{1}{k}.$$

Finally, we fix an increasing sequence $\{\mathcal{P}_k\}_{k=1}^\infty$ of finite partitions of X which generate $\mathcal{X}$.

Step 2 (Base case) Here we construct an initial partial speedup $(\mathbf{T}^1)^{\sigma_1}$ *of* $\mathbf{T}^\sigma$ *which is partially* $G-$*isomorphic to* $\mathbf{S}^\sigma$*.*

Consider the castle $\mathcal{C}_1^{\mathbf{S}}$ consisting of towers $\tau_{1,j}^{\mathbf{S}}$, each of size $[\mathbf{N}_1]$. For each $A_{1,j,\mathbf{v}}^{\mathbf{S}} \in \tau_{1,j}^{\mathbf{S}}$, let $A_{1,j,\mathbf{v}}^{\mathbf{T}}$ be a subset of X with $\mu(A_{1,j,\mathbf{v}}^{\mathbf{T}}) = \nu(A_{1,j,\mathbf{v}}^{\mathbf{S}})$ such that $\{A_{1,j,\mathbf{v}}^{\mathbf{T}}\}_{\mathbf{v} \in [\mathbf{N}_1]}$ forms a rectangular collection, denoted $\tau_{1,j}^{\mathbf{T}}$, with $|\tau_{1,j}^{\mathbf{T}}| \cap |\tau_{1,l}^{\mathbf{T}}| = \emptyset$ when $j \neq l$. For each j, because $\mu(A_{1,j,\mathbf{0}}^{\mathbf{T}}) = \nu(A_{1,j,\mathbf{0}}^{\mathbf{S}})$, we can find an isomorphism

$\phi_{1,j} : A^{\mathbf{T}}_{1,j,\mathbf{0}} \to A^{\mathbf{S}}_{1,j,\mathbf{0}}$. We will consider the collection $\{\tau^{\mathbf{T}}_{1,j}\}_j$, denoted $\mathcal{C}^{\mathbf{T}}_1$, to be a copy in X of $\mathcal{C}^{\mathbf{S}}_1$.

Use Corollary 3.10 for each j (with $\mathbf{m} = \mathbf{N}_1$ and $U_{\mathbf{v}} = B_{\epsilon_1}(e_G)\sigma_{\mathbf{S}}(\phi_{1,j}(x), \mathbf{v})$) to construct a $\mathbf{C}-$partial speedup $\mathbf{T}^{1,j}$ of $\mathbf{T}$ such that

(1) $\tau^{\mathbf{T}}_{1,j}$ is a speedup block for $\mathbf{T}^{1,j}$, and
(2) for every $\mathbf{v} \in [\mathbf{N_1}]$, for $\mu-$a.e. $x \in A^{\mathbf{T}}_{1,j,\mathbf{0}}$,

$$\sigma_{\mathbf{T}}\left(x, (\mathbf{T}^{1,j})_{\mathbf{v}}(x)\right)\left(\sigma_{\mathbf{S}}\left(\phi_{1,j}(x), \mathbf{v}\right)\right)^{-1} \in B_{\epsilon_1}(e_G). \tag{4.1}$$

Given $x \in A^{\mathbf{T}}_{1,j,\mathbf{v}}$ and $\mathbf{w} \in \mathbb{Z}^d$ such that $\mathbf{v} + \mathbf{w} \in [\mathbf{N}_1]$, define the cocycle associated to the speedup $\mathbf{T}^{1,j}$ to be

$$\sigma_{1,j}(x, \mathbf{w}) = \sigma_{\mathbf{T}}\left(x, (\mathbf{T}^{1,j})_{\mathbf{w}}(x)\right).$$

We can then define $\mathbf{T}^1$ so that $\mathbf{T}^1_{\mathbf{v}}$ coincides with $\mathbf{T}^{1,j}_{\mathbf{v}}$ wherever the latter map is defined; $\mathcal{C}^{\mathbf{T}}_1$ is therefore a speedup block for $\mathbf{T}^1$. We similarly define σ_1 so that $\sigma_1(x, \mathbf{v}) = \sigma_{1,j}(x, \mathbf{v})$ where the latter is defined.

What remains is to define the partial G-isomorphism between $(\mathbf{T}^1)^{\sigma_1}$ and $\mathbf{S}^{\sigma}$. We first extend, for each j, the isomorphism $\phi_{1,j}$ to the entire tower $|\tau^{\mathbf{T}}_{1,j}|$ so that for μ-a.e. $x \in A^{\mathbf{T}}_{1,j,\mathbf{0}}$ and each $\mathbf{v} \in [\mathbf{N_1}]$,

$$\phi_{1,j} \circ \mathbf{T}^{1,j}_{\mathbf{v}}(x) = \mathbf{S}_{\mathbf{v}} \circ \phi_{1,j}(x).$$

We next define $\alpha_{1,j} : |\tau^{\mathbf{T}}_{1,j}| \to G$ by setting for $x \in A^{\mathbf{T}}_{1,j,\mathbf{v}}$,

$$\alpha_{1,j}(x) = \sigma_{\mathbf{S}}(\phi_{1,j}(\mathbf{T}^{1,j}_{-\mathbf{v}}x), \mathbf{v})\, \sigma_{1,j}(\mathbf{T}^{1,j}_{-\mathbf{v}}x, \mathbf{v})^{-1}.$$

Similar in spirit to what we did before, define $\phi_1 : |\mathcal{C}^{\mathbf{T}}_1| \to |\mathcal{C}^{\mathbf{S}}_1|$ and $\alpha_1 : |\mathcal{C}^{\mathbf{T}}_1| \to G$ so that for each j they coincide with $\phi_{1,j}$ and $\alpha_{1,j}$, respectively, on $\tau^{\mathbf{T}}_{1,j}$. Set $\alpha_1(x) = e_G$ for $x \notin |C^{\mathbf{T}}_1|$.

We then have that the map

$$\Phi_1(x, g) = (\phi_1(x), \alpha_1(x)g)$$

is a $G-$isomorphism between $\left(\mathbf{T}^1\right)^{\sigma_1}$ and $\mathbf{S}^{\sigma}$ where these maps are thus far defined, i.e. for any $x \in A^{\mathbf{T}}_{1,j,\mathbf{v}}$ and any $\mathbf{w} \in \mathbb{Z}^d$ such that $\mathbf{v} + \mathbf{w} \in [\mathbf{N}_1]$, we have

$$\alpha_1\left(\mathbf{T}^1_{\mathbf{w}}(x)\right)\, \sigma_1(x, \mathbf{w})\, \left(\alpha_1(x)\right)^{-1} = \sigma_{\mathbf{S}}(\phi_1(x), \mathbf{w}).$$

Note that by the right-invariance of ρ and (4.1), we have for all $x \in \mathcal{C}^{\mathbf{T}}_1$,

$$\rho(\alpha_1(x), e_G) < \epsilon_1.$$

We complete the base case by setting $m_1 = 1$.

Step 3 (inductive step) Here we extend the partial speedup $(\mathbf{T}^k)^{\sigma_k}$ to another partial speedup $(\mathbf{T}^{k+1})^{\sigma_{k+1}}$ which is defined on more of X and is also partially $G-$isomorphic to $\mathbf{S}^{\sigma}$.

Assume that we have defined:

(1) numbers $1 = m_1 < m_2 < ... < m_k$, where for every $i \in \{2, ..., k\}$, we have

$$\sum_{n=m_i}^{\infty} 2\epsilon_n < \delta_i;$$

(2) $\mathbf{C}$−partial speedups $\mathbf{T}^1, \mathbf{T}^2, ..., \mathbf{T}^k$ of $\mathbf{T}$, defined on respective speedup blocks $\mathcal{C}_1^{\mathbf{T}}, \mathcal{C}_2^{\mathbf{T}}, \cdots, \mathcal{C}_k^{\mathbf{T}}$ of respective heights $\mathbf{N}_{m_1}, ..., \mathbf{N}_{m_k}$ such that

$$|\mathcal{C}_1^{\mathbf{T}}| \subseteq |\mathcal{C}_2^{\mathbf{T}}| \subseteq \cdots \subseteq |\mathcal{C}_k^{\mathbf{T}}|$$

and each $\mathbf{T}^{i+1}$ extends $\mathbf{T}^i$;

(3) isomorphisms $\phi_1, ..., \phi_k$, where each $\phi_i : |\mathcal{C}_i^{\mathbf{T}}| \to |\mathcal{C}_{m_i}^{\mathbf{S}}|$ satisfies

$$\phi_i \circ \mathbf{T}^i_{\mathbf{w}}(x) = \mathbf{S}_{\mathbf{w}} \circ \phi_i(x)$$

for all $x \in |\mathcal{C}_i^{\mathbf{T}}| \bigcap \mathbf{T}^i_{-\mathbf{w}}(|\mathcal{C}_i^{\mathbf{T}}|)$;

(4) corresponding cocycles $\sigma_1, ..., \sigma_k$ of the above partial speedups satisfying, for each i,

$$\sigma_i(x, \mathbf{w}) \left(\sigma_{\mathbf{S}}(\phi_i(x), \mathbf{w})\right)^{-1} \in B_{\epsilon_{m_i}}(e_G) \tag{4.2}$$

for a.e. $x \in \cup_j A^{\mathbf{T}}_{i,j,\mathbf{0}}$ and all $\mathbf{w} \in [\mathbf{N}_{m_i}]$; and

(5) transfer functions $\alpha_1, ..., \alpha_k : X \to G$ such that
- for all $x \in |\mathcal{C}_i^{\mathbf{T}}| \bigcap \mathbf{T}^i_{-\mathbf{w}}(|\mathcal{C}_i^{\mathbf{T}}|)$,

$$\alpha_i\left(\mathbf{T}^i_{\mathbf{w}}(x)\right) \sigma_i(x, \mathbf{w}) \left(\alpha_i(x)\right)^{-1} = \sigma_{\mathbf{S}}(\phi_i(x), \mathbf{w}); \tag{4.3}$$

- for every $x \in X$ and for every $i \in \{1, ..., k-1\}$,

$$\rho(\alpha_{i+1}(x), \alpha_i(x)) \leq 2\epsilon_{m_i}; \text{ and} \tag{4.4}$$

- $\rho(\alpha_i(x), e_G) < \epsilon$ for every $x \in X$.

Choose $m_{k+1} > m_k$ so that
(1) $\sum_{n=m_{k+1}}^{\infty} 2\epsilon_n < \delta_{k+1}$; and
(2) $\phi_k(\mathcal{P}_k)$ is approximated within distance $\frac{1}{2^{k+1}}$ (in the usual partition metric) by the levels of $\mathcal{C}^{\mathbf{S}}_{m_{k+1}}$.

In order to find our next speedup block $\mathcal{C}^{\mathbf{T}}_{k+1}$, note that we want it to be a copy in X of $\mathcal{C}^{\mathbf{S}}_{m_{k+1}}$. Recall that $|\mathcal{C}^{\mathbf{S}}_{m_k}| \subseteq |\mathcal{C}^{\mathbf{S}}_{m_{k+1}}|$ and $\mathcal{C}^{\mathbf{S}}_{m_{k+1}}$ consists of the towers $\tau^{\mathbf{S}}_{m_{k+1},j}$. For each level $A^{\mathbf{S}}_{m_{k+1},j,\mathbf{v}}$ contained in $|\mathcal{C}^{\mathbf{S}}_{m_k}|$, we define $A^{\mathbf{T}}_{k+1,j,\mathbf{v}} = \phi_k^{-1}\left(A^{\mathbf{S}}_{m_{k+1},j,\mathbf{v}}\right)$. Additional disjoint subsets of $X - \phi_k^{-1}(|\mathcal{C}^{\mathbf{S}}_{m_k}|)$ are arbitrarily chosen for the remaining levels $A^{\mathbf{T}}_{k+1,j,\mathbf{v}}$ so that $\mu(A^{\mathbf{T}}_{k+1,j,\mathbf{v}}) = \nu(A^{\mathbf{S}}_{m_{k+1},j,\mathbf{v}})$ for all j and $\mathbf{v}$. Thus for each j, $\{A^{\mathbf{T}}_{k+1,j,\mathbf{v}}\}_{\mathbf{v}\in[\mathbf{N}_{m_{k+1}}]}$ forms a rectangular collection which we denote by $\tau^{\mathbf{T}}_{k+1,j}$ and $|\tau^{\mathbf{T}}_{k+1,j}| \cap |\tau^{\mathbf{T}}_{k+1,l}| = \emptyset$ when $j \neq l$. By Lemma 2.8, $A^{\mathbf{S}}_{m_{k+1},j,\mathbf{0}}$ is disjoint from $\mathcal{C}^{\mathbf{S}}_{m_k}$ and thus ϕ_k is not defined on $A^{\mathbf{T}}_{k+1,j,\mathbf{0}}$: we let $\phi_{k+1,j} : A^{\mathbf{T}}_{k+1,j,\mathbf{0}} \to A^{\mathbf{S}}_{m_{k+1},j,\mathbf{0}}$ be an arbitrary isomorphism.

We then use Theorem 3.1 to construct our next $\mathbf{C}$−partial speedup. For the neighborhood of e_G, we use the closed ball $B_{\zeta_{k+1}}(e_G)$ where ζ_{k+1} is chosen so that $\zeta_{k+1} < \epsilon_{m_{k+1}}$ and (by the uniform continuity of group multiplication restricted to the compact set $K_{k+1} \times K_{k+1}$) if $a, a', b \in K_{k+1}$ satisfy $\rho(a, a') < \zeta_{k+1}$, then $\rho(ba, ba') < \epsilon_{m_k+1}$. Now fix j and let

$$R_j = \{\mathbf{v} \in [\mathbf{N}_{m_{k+1}}] : A^{\mathbf{T}}_{k+1,j,\mathbf{v}} \subseteq |\mathcal{C}^{\mathbf{T}}_k|\}.$$

Observe that $\bigcup_{\mathbf{v}\in R_j} A^{\mathbf{T}}_{k+1,j,\mathbf{v}}$ is the disjoint union of r (r is a finite number, possibly zero) speedup blocks $C_1, ..., C_r$ for $\mathbf{T}^k$. For $l = 1, ..., r$, we denote by $\mathbf{k}_l$ the initial vector of the speedup block C_l. We now use Theorem 3.1 (with $\mathbf{N} = \mathbf{N}_{m_{k+1}}$, $\mathbf{h} =$

$\mathbf{N}_{m_k}$, $A_{\mathbf{v}} = A^{\mathbf{T}}_{k+1,j,\mathbf{v}}$, $U = B_{\zeta_{k+1}}(e_G)$ and $g_{\mathbf{v}}(x) = \sigma_{\mathbf{S}}(\phi_{k+1,j}(x), \mathbf{v}))$ to construct a $\mathbf{C}-$partial speedup $\mathbf{T}^{k+1,j}$ of $\mathbf{T}$ such that

(1) $\{A^{\mathbf{T}}_{k+1,j,\mathbf{v}}\}_{\mathbf{v}\in[\mathbf{N}_{m_{k+1}}]}$ is a speedup block for $\mathbf{T}^{k+1,j}$;
(2) $\mathbf{T}^{k+1,j}$ extends $\mathbf{T}^k$; and
(3) for every $\mathbf{v} \in (\bigcup_{l=1}^r\{\mathbf{k}_l\}) \bigcup \left([\mathbf{N}_{m_{k+1}}] - \bigcup_l C_l\right)$, for $\mu-$a.e. $x \in A^{\mathbf{T}}_{k+1,j,\mathbf{0}}$,
$$\sigma_{\mathbf{T}}\left(x, (\mathbf{T}^{k+1,j})_{\mathbf{v}}(x)\right)\left(\sigma_{\mathbf{S}}\left(\phi_{k+1,j}(x), \mathbf{v}\right)\right)^{-1} \in B_{\zeta_{k+1}}(e_G).$$

We now extend $\phi_{k+1,j}$ to the entire tower $|\tau^{\mathbf{T}}_{k+1,j}|$ in the following way: for $y \in |\tau^{\mathbf{T}}_{k+1,j}|$, write $y = \mathbf{T}^{k+1,j}_{\mathbf{v}}(x)$ for some $x \in A^{\mathbf{T}}_{k+1,j,\mathbf{0}}$ and some $\mathbf{v}$. Then define
$$\phi_{k+1,j}(y) = \mathbf{S}_{\mathbf{v}}(\phi_{k+1,j}(x)).$$

After repeating the above procedure for each j, we set $\mathcal{C}^{\mathbf{T}}_{k+1}$ to be the union of the towers $\tau^{\mathbf{T}}_{k+1,j}$ and define $\mathbf{T}^{k+1}$ so that $\mathbf{T}^{k+1}_{\mathbf{v}}$ coincides with $\mathbf{T}^{k+1,j}_{\mathbf{v}}$ wherever the latter map is defined. Similarly, define $\phi_{k+1} : |\mathcal{C}^{\mathbf{T}}_{k+1}| \to |\mathcal{C}^{\mathbf{S}}_{m_{k+1}}|$ so that it coincides with each $\phi_{k+1,j}$ on $|\tau^{\mathbf{T}}_{k+1,j}|$. Also, let σ_{k+1} be the cocycle for $\mathbf{T}^{k+1}$, i.e. set
$$\sigma_{k+1}(x, \mathbf{w}) = \sigma_{\mathbf{T}}(x, \mathbf{T}^{k+1}_{\mathbf{w}}x)$$
for any $x \in |\mathcal{C}^{\mathbf{T}}_{k+1}| \bigcap \mathbf{T}^{k+1}_{-\mathbf{w}}(|\mathcal{C}^{\mathbf{T}}_{k+1}|)$.

All that remains is for us to define the transfer function $\alpha_{k+1} : X \to G$ and show it satisfies the stated properties. By our induction step, the transfer function $\alpha_k : X \to G$ relates $\sigma_k(x, \mathbf{v})$ and $\sigma_S(\phi_k(x), \mathbf{v})$. As $\phi_{k+1}(x)$ does not necessarily equal $\phi_k(x)$ even when both are defined, we cannot define α_{k+1} to simply extend α_k. So we first define a function $\overline{\alpha}_k$ which keeps track of the change from $\phi_k(x)$ to $\phi_{k+1}(x)$ by setting
$$\overline{\alpha}_k(x) = \begin{cases} \sigma_{\mathbf{S}}(\phi_{k+1}(x), -\mathbf{v})^{-1}\sigma_{\mathbf{S}}(\phi_k(x), -\mathbf{v}) & \text{if } x \in A^{\mathbf{T}}_{k,j,\mathbf{v}} \subseteq |\mathcal{C}^{\mathbf{T}}_k| \\ e_G & \text{if } x \notin |\mathcal{C}^{\mathbf{T}}_k| \end{cases}.$$
One can then check that for $x \in A^{\mathbf{T}}_{k,j,\mathbf{v}}$,
$$\overline{\alpha}_k(\mathbf{T}^k_{-\mathbf{v}}x)\,\alpha_k(\mathbf{T}^k_{-\mathbf{v}}x)\,\sigma_k(x, -\mathbf{v})\,\alpha_k(x)^{-1}\,\overline{\alpha}_k(x)^{-1} \;=\; \sigma_{\mathbf{S}}(\phi_{k+1}(x), -\mathbf{v}).$$
Since $\sigma_{k+1} = \sigma_k$ and $\mathbf{T}^{k+1} = \mathbf{T}^k$ where all are defined, the above can be written as
$$\sigma^{\overline{\alpha}_k\alpha_k}_{k+1}(x, -\mathbf{v}) \;=\; \sigma_{\mathbf{S}}(\phi_{k+1}(x), -\mathbf{v}) \text{ for } x \in A^{\mathbf{T}}_{k,j,\mathbf{v}}. \tag{4.5}$$
However, we need a transfer function which satisfies (4.3) for all $x \in \mathcal{C}^{\mathbf{T}}_{k+1}$ and all $\mathbf{u}$ such that $\mathbf{T}^{k+1}_{\mathbf{u}}x \in \mathcal{C}^{\mathbf{T}}_{k+1}$. In this more general case, the left side of (4.5) may or may not equal the right side. Thus we define a function $\widetilde{\alpha}_k : X \to G$ to keep track of this difference:
$$\widetilde{\alpha}_k(x) = \begin{cases} \sigma_{\mathbf{S}}(\phi_{k+1}(x), -\mathbf{w})^{-1}\sigma^{(\overline{\alpha}_k\alpha_k)}_{k+1}(x, -\mathbf{w}) & \text{if } x \in A^{\mathbf{T}}_{k+1,j,\mathbf{w}} \subseteq |\mathcal{C}^{\mathbf{T}}_{k+1}| \\ e_G & \text{if } x \notin |\mathcal{C}^{\mathbf{T}}_{k+1}| \end{cases}.$$
It can be shown that
$$\widetilde{\alpha}_k(\mathbf{T}^{k+1}_{-\mathbf{w}}x)\,\sigma^{\overline{\alpha}_k\alpha_k}_{k+1}(x, -\mathbf{w})\,\widetilde{\alpha}_k(x)^{-1} \;=\; \sigma_{\mathbf{S}}(\phi_{k+1}(x), -\mathbf{w}).$$

In other words, by setting

$$\alpha_{k+1}(x) \;=\; \widetilde{\alpha}_k(x)\,\overline{\alpha}_k(x)\,\alpha_k(x),$$

we obtain

$$\alpha_{k+1}(\mathbf{T}^{k+1}_{-\mathbf{w}}x)\,\sigma_{k+1}(x,-\mathbf{w})\alpha_{k+1}(x)^{-1} = \sigma_{\mathbf{S}}(\phi_{k+1}(x),-\mathbf{w}). \tag{4.6}$$

Equation (4.6) has $\mathbf{w}$ where $x \in A^{\mathbf{T}}_{k+1,j,\mathbf{w}}$. Now let $\mathbf{u}$ be such that $\mathbf{T}^{k+1}_{\mathbf{u}}x \in \mathcal{C}^{\mathbf{T}}_{k+1}$: then $\mathbf{T}^{k+1}_{\mathbf{u}}x \in A^{\mathbf{T}}_{k+1,j,\mathbf{w}+\mathbf{u}}$ and we also know

$$\begin{aligned}\alpha_{k+1}(\mathbf{T}^{k+1}_{-\mathbf{w}}x)\,\sigma_{k+1}(\mathbf{T}^{k+1}_{\mathbf{u}}x,-(\mathbf{w}+\mathbf{u}))\,\alpha_{k+1}(\mathbf{T}^{k+1}_{\mathbf{u}}x)^{-1} &= \\ \sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}^{k+1}_{\mathbf{u}}x),-(\mathbf{w}+\mathbf{u})).\end{aligned} \tag{4.7}$$

By the cocycle equation,

$$\begin{aligned}\sigma_{\mathbf{S}}(\phi_{k+1}(x),\mathbf{u}) \;&=\; \sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}^{k+1}_{-\mathbf{w}}x),\mathbf{w}+\mathbf{u})\,\sigma_{\mathbf{S}}(\phi_{k+1}(x),-\mathbf{w}) \\ &= (\sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}^{k+1}_{\mathbf{u}}x),-(\mathbf{w}+\mathbf{u})))^{-1}\,\sigma_{\mathbf{S}}(\phi_{k+1}(x),-\mathbf{w}).\end{aligned}$$

Plugging in (4.6) and (4.7), this reduces to

$$\sigma_{\mathbf{S}}(\phi_{k+1}(x),\mathbf{u}) \;=\; \alpha_{k+1}(\mathbf{T}^{k+1}_{\mathbf{u}}x)\,\sigma_{k+1}(x,\mathbf{u})\,\alpha_{k+1}(x)^{-1}$$

which shows our α_{k+1} satisfies condition (4.3).

For condition (4.4), rewrite $\rho(\alpha_{k+1}(x),\alpha_k(x))$ as

$$\rho(\widetilde{\alpha}_k(x)\,\overline{\alpha}_k(x)\,\alpha_k(x),\,\alpha_k(x)) = \rho(\widetilde{\alpha}_k(x),\,\overline{\alpha}_k(x)^{-1}) \le \rho(\widetilde{\alpha}_k(x),\,e_G)+\rho(e_G,\,\overline{\alpha}_k(x)^{-1}).$$

Consider first $\rho(e_G,\,\overline{\alpha}_k(x)^{-1}) = \rho(\overline{\alpha}_k(x),e_G)$. For $x \in A^{\mathbf{T}}_{k,j,\mathbf{v}}$, this equals

$$\begin{aligned}&\rho(\sigma_{\mathbf{S}}(\phi_{k+1}(x),-\mathbf{v})^{-1}\sigma_{\mathbf{S}}(\phi_k(x),-\mathbf{v}),e_G) \\ &= \rho(\sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}^k_{-\mathbf{v}}x),\mathbf{v})\,\sigma_{\mathbf{S}}(\phi_k(\mathbf{T}^k_{-\mathbf{v}}x),\mathbf{v})^{-1},\,e_G).\end{aligned}$$

Although $\phi_k(x)$ is not necessarily equal to $\phi_{k+1}(x)$, they are both on the same level in $\mathcal{C}^{\mathbf{S}}_{m_k}$ and by Lemma 2.8, both $\sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}^k_{-\mathbf{v}}x),\mathbf{v})$ and $\sigma_{\mathbf{S}}(\phi_k(\mathbf{T}^k_{-\mathbf{v}}x),\mathbf{v})$ lie in $U_{m_k}g_{\mathbf{v}}$. Thus

$$\sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}^k_{-\mathbf{v}}x),\mathbf{v})\,\sigma_{\mathbf{S}}(\phi_k(\mathbf{T}^k_{-\mathbf{v}}x),\mathbf{v})^{-1} \in U_{m_k}U_{m_k}^{-1} \subseteq B_{\epsilon_{m_k}}(e_G)$$

and we have

$$\rho(e_G,\,\overline{\alpha}_k(x)^{-1}) \le \epsilon_{m_k}.$$

Note that if $x \notin \mathcal{C}^{\mathbf{T}}_k$, then $\overline{\alpha}_k(x) = e_G$ and the above holds trivially.

For $\rho(\widetilde{\alpha}_k(x),\,e_G)$, we have two cases: in the first, $x \in |\mathcal{C}^{\mathbf{T}}_{k+1}| - |\mathcal{C}^{\mathbf{T}}_k|$ or x is in the base of $|\mathcal{C}^{\mathbf{T}}_k|$, and in the second x is in some $A^{\mathbf{T}}_{k+1,j,\mathbf{w}} \cap A^{\mathbf{T}}_{k,l,\mathbf{v}}$ with $\mathbf{v} \neq \mathbf{0}$. In the first case, $\overline{\alpha}_k(x) = \alpha_k(x) = e_G$ and

$$\begin{aligned}\widetilde{\alpha}_k(x) &= (\sigma_{\mathbf{S}}(\phi_{k+1}(x),-\mathbf{w}))^{-1}\sigma_{k+1}(x,-\mathbf{w}) \\ &= \sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}^{k+1}_{-\mathbf{w}}x),\mathbf{w})\sigma_{k+1}(\mathbf{T}^{k+1}_{-\mathbf{w}}x,\mathbf{w})^{-1},\end{aligned}$$

where $x \in A^{\mathbf{T}}_{k+1,j,\mathbf{w}}$. We know by Theorem 3.1 that

$$\sigma_{k+1}(\mathbf{T}^{k+1}_{-\mathbf{w}}x,\mathbf{w})\,\sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}^{k+1}_{-\mathbf{w}}x),\mathbf{w})^{-1} \in B_{\zeta_{k+1}}(e_G),$$

so therefore $\rho(\sigma_{k+1}(\mathbf{T}_{-\mathbf{w}}^{k+1}x, \mathbf{w})\, \sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}_{-\mathbf{w}}^{k+1}x), \mathbf{w})^{-1}, e_G) < \zeta_{k+1}$. But

$$\begin{aligned}\rho(\sigma_{k+1}(\mathbf{T}_{-\mathbf{w}}^{k+1}x, \mathbf{w})\sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}_{-\mathbf{w}}^{k+1}x), \mathbf{w})^{-1}, e_G) &= \\ &\rho(e_G, \sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}_{-\mathbf{w}}^{k+1}x), \mathbf{w})\sigma_{k+1}(\mathbf{T}_{-\mathbf{w}}^{k+1}x, \mathbf{w})^{-1})\end{aligned}$$

and we end up with $\rho(\widetilde{\alpha}_k(x), e_G) < \zeta_{k+1} < \epsilon_{m_{k+1}}$.

In the second case, $\rho(\widetilde{\alpha}_k(x), e_G) = \rho(\sigma_{\mathbf{S}}(\phi_{k+1}(x), -\mathbf{w})^{-1},\ \sigma_{k+1}^{\overline{\alpha}_k\alpha_k}(x, -\mathbf{w})^{-1})$ which equals $\rho(\sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}_{-\mathbf{w}}^{k+1}x), \mathbf{w}),\ \sigma_{k+1}^{\overline{\alpha}_k\alpha_k}(\mathbf{T}_{-\mathbf{w}}^{k+1}x, \mathbf{w})\,)$. Using the cocycle equation, we relate the position of x and $\phi_{k+1}(x)$ in their $(k+1)$-tower to their location in the k-tower and the k-tower's location in the $(k+1)$-tower, i.e.

$$\sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}_{-\mathbf{w}}^{k+1}x), \mathbf{w}) \;=\; \sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}_{-\mathbf{v}}^{k+1}x), \mathbf{v})\, \sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}_{-\mathbf{w}}^{k+1}x), \mathbf{w}-\mathbf{v})$$

and

$$\sigma_{k+1}^{\overline{\alpha}_k\alpha_k}(\mathbf{T}_{-\mathbf{w}}^{k+1}x, \mathbf{w}) \;=\; \sigma_{k+1}^{\overline{\alpha}_k\alpha_k}(\mathbf{T}_{-\mathbf{v}}^{k+1}x, \mathbf{v})\, \sigma_{k+1}^{\overline{\alpha}_k\alpha_k}(\mathbf{T}_{-\mathbf{w}}^{k+1}x, \mathbf{w}-\mathbf{v}).$$

Note that the first terms of the right-hand sides are equal by (4.5) and

$$\rho(\sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}_{-\mathbf{w}}^{k+1}x)), \mathbf{w}-\mathbf{v}),\ \sigma_{k+1}^{\overline{\alpha}_k\alpha_k}(\mathbf{T}_{-\mathbf{w}}^{k+1}x, \mathbf{w}-\mathbf{v}) < \zeta_{k+1}$$

by the argument used in the first case. We thus have that $\rho(\widetilde{\alpha}_k(x), e_G)$ has the form $\rho(ba, ba')$ with $\rho(a, a') < \zeta_{k+1}$ The result then follows from the definition of ζ_{k+1} once we know all the terms are elements of K_{k+1}. We first note that $\sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}_{-\mathbf{w}}^{k+1}x), \mathbf{w}-\mathbf{v}) \in \widetilde{K}_{k+1} \subset K_{k+1}$ by definition. By rewriting

$$\sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}_{-\mathbf{v}}^{k+1}x), \mathbf{v}) \;=\; \sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}_{-\mathbf{w}}^{k+1}x), \mathbf{w})\, [\sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}_{-\mathbf{w}}^{k+1}x), \mathbf{w}-\mathbf{v})]^{-1},$$

we see that

$$\sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}_{-\mathbf{v}}^{k+1}x), \mathbf{v}) \in \widetilde{K}_{k+1}(\widetilde{K}_{k+1})^{-1} \subset K_{k+1}.$$

Finally, since $\sigma_{k+1}^{\overline{\alpha}_k\alpha_k}(T_{-\mathbf{w}}^{k+1}x, \mathbf{w}-\mathbf{v})$ is within ζ_{k+1} of $\sigma_{\mathbf{S}}(\phi_{k+1}(\mathbf{T}_{-\mathbf{w}}^{k+1}x), \mathbf{w}-\mathbf{v}) \in \widetilde{K}_{k+1}$, we have

$$\sigma_{k+1}^{\overline{\alpha}_k\alpha_k}(T_{-\mathbf{w}}^{k+1}x, \mathbf{w}-\mathbf{v}) \in B_{\zeta_{k+1}}(e_G)\widetilde{K}_{k+1} \subset B_{\epsilon}(e_G)\widetilde{K}_{k+1} \subset K_{k+1},$$

as wanted.

We can then say say our transfer function satisfies condition (4.4) by noting

$$\rho\left(\alpha_{k+1}(x), \alpha_k(x)\right) \leq \epsilon_{m_{k+1}} + \epsilon_{m_k} < 2\epsilon_{m_k}.$$

For the last criterion on α_k, note

$$\begin{aligned}\rho\left(\alpha_{k+1}(x), e_G\right) &\leq \rho\left(\alpha_{k+1}(x), \alpha_k(x)\right) + \ldots + \rho\left(\alpha_1(x), e_G\right) \\ &< 2\epsilon_{m_k} + 2\epsilon_{m_{k-1}} + \ldots + 2\epsilon_{m_1} + \epsilon_1 \\ &< \epsilon.\end{aligned}$$

This completes the inductive step.

Step 4: Define the speedup $\overline{\mathbf{T}}$. After repeating the procedure in Step 3 indefinitely, we obtain a sequence of castles $\mathcal{C}_k^{\mathbf{T}}$ in X for $\mathbf{C}-$partial speedups $\mathbf{T}^k$ of $\mathbf{T}$, where

(1) the levels of $\mathcal{C}_k^{\mathbf{T}}$ approximate the partition $\mathcal{P}_{k-1}$ to within $\frac{1}{2^k}$;
(2) each $\mathcal{C}_k^{\mathbf{T}}$ is a speedup block for $\mathbf{T}^k$;
(3) each $\mathbf{T}^{k+1}$ extends $\mathbf{T}^k$;
(4) for each castle $\mathcal{C}_k^{\mathbf{T}}$, there is an isomorphism $\phi_k : |\mathcal{C}_k^{\mathbf{T}}| \to |\mathcal{C}_{m_k}^{\mathbf{S}}|$ intertwining $\mathbf{T}^k$ and $\mathbf{S}$; and

(5) for each castle $\mathcal{C}_k^{\mathbf{T}}$, there is a function $\alpha_k : X \to G$ so that
   - the map $\Phi_k : (x, g) \mapsto (\phi_k(x), \alpha_k(x)g)$ is a G−isomorphism between $(\mathbf{T}^k)^{\sigma_k}$ and $\mathbf{S}^{\sigma}$,
   - $\rho(\alpha_k(x), \alpha_{k+1}(x)) \leq 2\epsilon_{m_k}$, and
   - $\rho(\alpha_k(x), e_G) < \epsilon$.

We can then define the $\mathbf{C}$-speedup to be $\overline{\mathbf{T}} = \lim_{k\to\infty} \mathbf{T}^k$. We define its cocycle $\overline{\sigma}$ by setting $\overline{\sigma}(x, \mathbf{v}) = \sigma_k(x, \mathbf{v})$ where k is large enough so that x and $\overline{\mathbf{T}}_{\mathbf{v}}(x)$ lie in $\mathcal{C}_k^{\mathbf{T}}$. Since ρ is a complete metric, we see that the sequence $\{\alpha_k\}$ converges uniformly to a function $\overline{\alpha} : X \to G$ which satisfies $\rho(\overline{\alpha}(x), e_G) \leq \epsilon$ for all $x \in X$. Note that by our construction, each ϕ_{k+1} agrees (setwise, but not necessarily pointwise) with ϕ_k on the levels of $\mathcal{C}_k^{\mathbf{T}}$. Since these levels increase to the full σ−algebra $\mathcal{X}$, the maps ϕ_k determine an isomorphism ϕ between $\overline{\mathbf{T}}$ and $\mathbf{S}$ which, for each k, agrees setwise with ϕ_k on the levels of $\mathcal{C}_k^{\mathbf{T}}$.

Finally, we explain why the map $\Phi : X \times G \to Y \times G$ defined by $\Phi(x, g) = (\phi(x), \overline{\alpha}(x)g)$ is a G−isomorphism between $\overline{\mathbf{T}}^{\overline{\sigma}}$ and $\mathbf{S}^{\sigma}$. Fix $\mathbf{v} \in \mathbb{Z}^d$ and note that for a.e. x, we can find K such that for all $k \geq K$, $\overline{\mathbf{T}}_{\mathbf{v}}(x) = \mathbf{T}^k_{\mathbf{v}}(x)$. It is sufficient to show that

$$\rho(\overline{\sigma}^{\overline{\alpha}}(x, \mathbf{v}),\, \sigma_{\mathbf{S}}(\phi(x), \mathbf{v}))$$

is arbitrarily small.

By the triangle inequality we see

$$\rho(\overline{\sigma}^{\overline{\alpha}}(x, \mathbf{v}),\, \sigma_{\mathbf{S}}(\phi(x), \mathbf{v})) \leq \rho(\overline{\sigma}^{\overline{\alpha}}(x, \mathbf{v}), \sigma_{\mathbf{S}}(\phi_k(x), \mathbf{v})) + \rho(\sigma_{\mathbf{S}}(\phi_k(x), \mathbf{v}), \sigma_{\mathbf{S}}(\phi(x), \mathbf{v})).$$

Consider the first term. Recall that $\mathbf{S}$ is isomorphic to $\mathbf{T}_k^{\sigma_k}$ on the appropriate domain, so we know

$$\begin{aligned}\sigma_{\mathbf{S}}(\phi_k(x), \mathbf{v}) = \sigma_k^{\alpha_k}(x, \mathbf{v}) &= \alpha_k(\mathbf{T}^k_{\mathbf{v}} x)\, \sigma_k(x, \mathbf{v})\, \alpha_k(x)^{-1} \\ &= \alpha_k(\overline{\mathbf{T}}_{\mathbf{v}} x)\, \overline{\sigma}(x, \mathbf{v})\, \alpha_k(x)^{-1}\end{aligned}$$

and thus

$$\begin{aligned}&\rho(\overline{\sigma}^{\overline{\alpha}}(x, \mathbf{v}), \sigma_{\mathbf{S}}(\phi_k(x), \mathbf{v})) \\ &\quad = \rho(\overline{\alpha}(\overline{\mathbf{T}}_{\mathbf{v}} x)\, \overline{\sigma}(x, \mathbf{v})\, \overline{\alpha}(x)^{-1},\ \alpha_k(\overline{\mathbf{T}}_{\mathbf{v}} x)\, \overline{\sigma}(x, \mathbf{v})\, \alpha_k(x)^{-1}).\end{aligned} \tag{4.8}$$

But we know that for all $z \in X$,

$$\rho(\alpha_k(z), \overline{\alpha}(z)) \leq \rho(\alpha_k(z), \alpha_{k+1}(z)) + \rho(\alpha_{k+1}(z), \alpha_{k+2}(z)) + ... \leq \sum_{i=k}^{\infty} 2\epsilon_{m_i} < \delta_k.$$

Thus (4.8) has the form $\rho(h_1 g\, h_3, h_2 g\, h_4)$ where $\rho(h_1, h_2) < \delta_k$ and $\rho(h_3, h_4) < \delta_k$. Once we have that $\overline{\sigma}(x, \mathbf{v}) \in K_k$, the choice of δ_k made in Step 1 gives us that (4.8) is less than $\frac{1}{k}$.

To show that $\overline{\sigma}(x, \mathbf{v}) \in K_k$, let $\mathbf{w}$ be such that $x \in A^{\mathbf{T}}_{k,j,\mathbf{w}}$. Using the cocycle condition and that $\overline{\sigma}(x, \mathbf{v}) = \sigma_k(x, \mathbf{v})$ here, we have

$$\sigma_k(x, \mathbf{v}) = \sigma_k(\mathbf{T}^k_{-\mathbf{w}} x, \mathbf{v} + \mathbf{w})\, \sigma_k(\mathbf{T}^k_{-\mathbf{w}} x, \mathbf{w})^{-1}.$$

We then use that $\sigma_{\mathbf{S}} = \sigma_k^{\alpha_k}$ can be written $\sigma_{\mathbf{S}}^{\alpha_k^{-1}} = \sigma_k$ to write $\sigma_k(x, \mathbf{v})$ as

$$\begin{aligned}&\alpha_k(\mathbf{T}^k_{\mathbf{v}} x)^{-1} \sigma_{\mathbf{S}}(\phi_k(\mathbf{T}^k_{-\mathbf{w}} x), \mathbf{v}{+}\mathbf{w}) \alpha_k(\mathbf{T}^k_{-\mathbf{w}} x)\ [\alpha_k(x)^{-1} \sigma_{\mathbf{S}}(\phi_k(\mathbf{T}^k_{-\mathbf{w}} x), \mathbf{w}) \alpha_k(\mathbf{T}^k_{-\mathbf{w}} x)]^{-1} \\ &\quad = \alpha_k(\mathbf{T}^k_{\mathbf{v}} x)^{-1} \sigma_{\mathbf{S}}(\phi_k(\mathbf{T}^k_{-\mathbf{w}} x), \mathbf{v} + \mathbf{w})(\sigma_{\mathbf{S}}(\phi(\mathbf{T}^k_{-\mathbf{w}} x), \mathbf{w}))^{-1} \alpha_k(x)\end{aligned}$$

which is in $B_\epsilon(e_G)\, \widetilde{K}_k\, \widetilde{K}_k^{-1}\, B_\epsilon(e_G) = K_k$, as wanted.

Now we consider the second term, $\rho(\sigma_{\mathbf{S}}(\phi(x),\mathbf{v}),\sigma_{\mathbf{S}}(\phi_k(x),\mathbf{v}))$. We know $\phi(x)$ and $\phi_k(x)$ lie on the same level of $C^{\mathbf{S}}_{m_k}$; call the height of that level $\mathbf{w}$. Let $z = \mathbf{S}_{-\mathbf{w}}(\phi(x))$ and $z_k = \mathbf{S}_{-\mathbf{w}}(\phi_k(x))$. Then

$$\begin{aligned}&\rho(\sigma_{\mathbf{S}}(\phi(x),\mathbf{v}),\sigma_{\mathbf{S}}(\phi_k(x),\mathbf{v})\,)\\&=\rho(\sigma_{\mathbf{S}}(z,\mathbf{v}+\mathbf{w})\,\sigma_{\mathbf{S}}(z,\mathbf{w})^{-1},\sigma_{\mathbf{S}}(z_k,\mathbf{v}+\mathbf{w})\,\sigma_{\mathbf{S}}(z_k,\mathbf{w})^{-1})\\&=\rho(\sigma_{\mathbf{S}}(z,\mathbf{v}+\mathbf{w})\,e_G\,\sigma_{\mathbf{S}}(z,\mathbf{w})^{-1},\sigma_{\mathbf{S}}(z_k,\mathbf{v}+\mathbf{w})\,e_G\,\sigma_{\mathbf{S}}(z_k,\mathbf{w})^{-1}).\end{aligned}$$

Recall the castles for $\mathbf{S}$ were chosen so that for each tower and each level $\mathbf{v}$ at the k^{th} step, there is a vector $g_{\mathbf{v}}$ such that $\sigma_{\mathbf{S}}(z,\mathbf{v}) \in U_k g_{\mathbf{v}}$ for all z in the base of that tower. Thus we have that $\sigma_{\mathbf{S}}(z,\mathbf{v}+\mathbf{w})$ and $\sigma_{\mathbf{S}}(z_k,\mathbf{v}+\mathbf{w})$ both belong to $U_{m_k} g_{\mathbf{v}+\mathbf{w}}$, i.e.

$$\sigma_{\mathbf{S}}(z,\mathbf{v}+\mathbf{w})\,\sigma_{\mathbf{S}}(z_k,\mathbf{v}+\mathbf{w})^{-1} \in U_{m_k}\,U_{m_k}^{-1} \subseteq B_{\epsilon_{m_k}}(e_G).$$

By our choice of m_k made at the beginning of Step 3, we have $\rho(\sigma_{\mathbf{S}}(z,\mathbf{v}+\mathbf{w}),\ \rho(\sigma_{\mathbf{S}}(z_k,\mathbf{v}+\mathbf{w})\,) < \delta_k$ and similarly $\rho(\sigma_{\mathbf{S}}(z,\mathbf{w}),\ \sigma_{\mathbf{S}}(z_k,\mathbf{w})\ ) < \delta_k$. Again we use our choice of δ_k to conclude $\rho(\sigma_{\mathbf{S}}(\phi(x),\mathbf{v}),\sigma_{\mathbf{S}}(\phi_k(x),\mathbf{v})\,) < \frac{1}{k}$.

Putting these two terms together yields

$$\rho(\overline{\sigma}^{\overline{\alpha}}(x,\mathbf{v}),\sigma_{\mathbf{S}}(\phi(x),\mathbf{v})) \leq \frac{1}{k}+\frac{1}{k}=\frac{2}{k}.$$

Since k is arbitrary, we get $\sigma_{\mathbf{S}}(\phi(x),\mathbf{v}) = \overline{\sigma}^{\overline{\alpha}}(x,\mathbf{v})$ for a.e. x and all $\mathbf{v}$ and thus $\mathbf{S}^{\sigma} = \overline{\mathbf{T}}^{\overline{\sigma}}$, as desired.

□

Theorem 1.1 asserts that the transfer function can be restricted to take values in any predetermined neighborhood of the identity element of G. If G is a discrete group, then such a neighborhood can be choosen to consist of only the identity element itself, and we immediately get the following stronger result:

Corollary 4.1. *Fix a finite or countable group G, and let $(X,\mathcal{X},\mu,\mathbf{T})$ and $(Y,\mathcal{Y},\nu,\mathbf{S})$ be $\mathbb{Z}^d-$actions with $\mathbf{S}$ aperiodic. Set $\mathbf{T}^{\sigma}$ and $\mathbf{S}^{\sigma}$ to be $G-$extensions of $\mathbf{T}$ and $\mathbf{S}$, respectively. Let $\mathbf{C} \subseteq \mathbb{Z}^d$ be any cone. Suppose $\mathbf{T}^{\sigma}$ is ergodic.*

Then there is a speedup $\overline{\mathbf{T}}^{\overline{\sigma}}$ of $\mathbf{T}^{\sigma}$ for which the speedup function is measurable with respect to $\mathcal{X}$ and takes values only in $\mathbf{C}$, such that $\overline{\mathbf{T}}^{\overline{\sigma}}$ is $G-$isomorphic to $\mathbf{S}^{\sigma}$, via a $G-$isomorphism whose transfer function $\overline{\alpha}$ satisfies $\overline{\alpha}(x) = e_G$ a.e.

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