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Commuting endomorphisms of the circle

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Abstract. In this paper the results of Shub and Sacksteder are extended to the following theorem: let f_1 and f_2 be two commuting, expansive, orientation-preserving maps of the circle with a common fixed point and with both in $C^{1+\varepsilon}$ or C^r, $r \geq 2$. Assume f_1 is p-to-1 and f_2 is q-to-1 where p and q generate a nonlacunary semigroup. Then there exists a diffeomorphism g of the same class such that $gf_1g^{-1} = T_p$ and $gf_2g^{-1} = T_q$.

1. *Introduction*

Consider the circle to be the group $[0, 1)$ under addition modulo 1. By T_p we will mean the function $T_p(x) = px \bmod 1$. This paper extends the following results.

THEOREM. [Sh]. *Let $f : S^1 \to S^1$ be a C^1 function such that $|df| > 1$ on all of S^1, where df is the derivative of f. If f is a p-to-1 map then f is homeomorphically conjugate to T_p, i.e. there exists a homeomorphism g with $g^{-1}fg = T_p$.*

THEOREM. [S]. *Let f_1 and f_2 be expanding, orientation-preserving maps of S^1 that commute, have degrees n and m respectively, and are of class C^{n-} $(n \geq 2)$. Then if n and m are relatively prime, there is a diffeomorphism g of class C^{n-} satisfying* $g(f_1(x)) = T_n g(x)$ and $g(f_2(x)) = T_m g(x)$.

There is a well known obstruction to Shub's homeomorphism g being a diffeomorphism. We describe this in § 2. However, we can now extend Sacksteder's result to integers p and q where p and q generate a nonlacunary semigroup of $\mathbb{N}$. In § 3 this theorem is stated formally along with the definition of nonlacunary. §§ 4 and 5 contain the results needed to prove this theorem. In § 6 we will discuss the case when f_1 and f_2 do not have a common fixed point.

2. *The diffeomorphism obstruction*

From Shub's theorem we know $f = gT_pg^{-1}$. So g must send the fixed points of T_p to those of f. Let us write this as $g(i/(p-1)) = z_i$. If g is differentiable then

$$df(x) = dg(T_pg^{-1}(x))\, dT_p(g^{-1}(x))\, dg^{-1}(x).$$

In particular,

$$df(z_i) = dg(i/(p-1))\, dT_p(i/(p-1))\, dg^{-1}(z_i).$$

But $dg(i/(p-1))\times dg^{-1}(z_i)=1$, so

$$df(z_i)=dT_p(i/p-1))=p.$$

Thus g cannot be differentiable if f has a derivative at a fixed point not equal to p. Applying this idea to powers of f gives obstructions from all periodic orbits of f. According to Katok [**Ka**] these are the only obstructions, and an alternative proof of our main theorem can be constructed along these lines.

3. *Main theorem*

Definition. A semigroup of $\mathbb{N}$ is said to be nonlacunary if it is not contained in a singly generated semigroup. For instance, the multiplicative semigroup generated by 6 and 10 is nonlacunary, the one generated by 4 and 8 is not.

Recall that $f\in C^{1+\varepsilon}$ if f has a continuous derivative that is Hölder, with Hölder constant ε. We say $f\in C^r$, $r\geq 2$, if f has r continuous derivatives. By $\mathscr{C}$ let us mean one of these classes of functions. If $\mathscr{C}=C^{1+\varepsilon}$ then by $\mathscr{C}'$ we mean C^ε. If $\mathscr{C}=C^r$ then $\mathscr{C}'=C^{r-1}$.

MAIN THEOREM *Let f_1 and f_2 be two commuting, orientation-preserving maps in $\mathscr{C}$ with a common fixed point. Assume they are both expanding and that f_1 is p-to-1 and f_2 is q-to-1 where p and q generate a nonlacunary semigroup. Then there exists a diffeomorphism $g\in\mathscr{C}$ such that $gf_1g^{-1}=T_p$ and $gf_2g^{-1}=T_q$.*

PROPOSITION 1. *For f_1 and f_2 as in the theorem, there exists a homeomorphism g that conjugates f_1 to T_p and f_2 to T_q.*

Proof. Let 0 be the common fixed point. From Shub there is a homeomorphism $g:S^1\to S^1$ with $g^{-1}f_1g=T_p$. Since T_p and f_1 both have 0 as a fixed point we can assume g fixes 0 (compose g with a rotation as necessary). Then $g^{-1}f_2g=\tilde{f}_2$ is a q-to-1 orientation-preserving map of S^1 to itself, fixing 0 and commuting with T_p.

Let $\hat{f}_2$ be the unique lift of $\tilde{f}_2$ to a homeomorphism of the universal cover, $\mathbb{R}$, fixing 0. Thus $\hat{f}_2(px)=p\hat{f}_2(x)$ and for $n\in\mathbb{N}$, $\hat{f}_2(n)=qn$. It is then easy to show that $f_2(\sum a_j/p^j)=q(\sum a_j/p^j)$ and hence $\hat{f}_2(x)=qx$, thus $\tilde{f}_2=T_q$.

We want to show that this conjugation g is in $\mathscr{C}$. This will be done in the next two sections. First we generalize our theorem to the non-orientation preserving case.

COROLLARY OF MAIN THEOREM. *Let f_1 and f_2 be as described in the main theorem except not necessarily orientation preserving. Then there exists $g\in\mathscr{C}$ such that $gf_1g^{-1}=T_r$ and $gf_2g^{-1}=T_s$ with $|r|=p$ and $|s|=q$. The sign of r and s is determined by the orientation of the associated map.*

Proof. The set of continuous, monotone maps that commute with some T_m is $\{T_n+k/(m-1):k=0,\ldots,m-2 \text{ and } n\in\mathbb{Z}\}$. This follows from an argument similar to the proof of Proposition 1 and the fact that any map that commutes with T_m can only permute its fixed points. Now we use the main theorem to find $g\in\mathscr{C}$ such that $gf_1g^{-1}=T_{p^2}$ and $gf_2g^{-1}=T_{q^2}$. As before, we can assume g fixes 0. Then gf_1g^{-1} commutes with T_{p^2} so must have the form $T_n+k/(p^2-1)$. Since it fixes zero and has degree p it must equal T_r with $|r|=p$. Similarly for gf_2g^{-1}. □

4. *Preliminary results*

In this section we will define a measure μ and show that it is f_1 and f_2 invariant. Using the structure of Parry and Pollicott [PP] and Krzyzewski [Kr] on the operator introduced by Ruelle [R], define $L: C(S^1) \to C(S^1)$ by $L(w)(x) = \sum_{y: f_1 y = x} w(y)/df_1(y)$. In Parry's notation, this is $L_{(-\log df_1)}$. If we let $\tilde{f}_1$ be the lift of f_1 so that $\tilde{f}_1: [0, 1) \to [0, p)$ is 1-1 then we can rewrite this operator as

$$L(w)(x) = \sum_{i=0}^{p-1} \frac{w(\tilde{f}_1^{-1}(x+i))}{d\tilde{f}_1(\tilde{f}_1^{-1}(x+i))}.$$

From [Kr] we get the following.

PROPOSITION 2. *There exists a strictly positive eigenfunction $h \in \mathscr{C}'$, corresponding to eigenvalue* 1, *of the operator L.*

Thus we have $L(h) = h$.

PROPOSITION 3.

$$\int_A h(x)\, dx = \int_{f_1^{-1}A} h(x)\, dx.$$

Proof. We have

$$\int_A h(x)\, dx = \int_A Lh(x)\, dx = \int_A \sum_{i=0}^{p-1} \frac{h(\tilde{f}_1^{-1}(x+i))}{d\tilde{f}_1(\tilde{f}_1^{-1}(x+i))}\, dx$$

$$= \int_{A \cup A+1 \cup \cdots \cup A+(p-1)} \frac{h(\tilde{f}_1^{-1}(z))}{d\tilde{f}_1(\tilde{f}_1^{-1}(z))}\, dz.$$

Now use the change of variables given by $y = \tilde{f}_1^{-1}(z)$. Then $dy = dz/d\tilde{f}_1(\tilde{f}_1^{-1}(z))$ and we get $\int_{f_1^{-1}A} h(y)\, dy$ as wanted. □

Definition 4. Take μ to be the measure given by $d\mu = h(x)\, dx$.

By Proposition 3, μ is f_1 invariant. We next want to show that it is also f_2 invariant.

PROPOSITION 5. (See [Kr].) *The system (S^1, f_1, μ) is exact. In particular, it is ergodic and $h_\mu(f_1) > 0$.*

Let $f_2^*\mu$ be the measure defined by $f_2^*\mu(E) = \mu(f_2^{-1}E)$. Then $f_2^*\mu$ is f_1 invariant, since f_1 and f_2 commute.

PROPOSITION 6. $f_2^*\mu = \mu$.

Proof. As f_2 is differentiable,

$$\frac{df_2^*\mu}{d\mu}(x) = \sum_{y: f_2(y) = x} \frac{h(y)}{h(x)\, df_2(y)} > 0.$$

Since μ is ergodic for f_1, this gives us the result. □

This tells us that μ is invariant for f_2.

5. *Completion of result*

Consider $g^*\mu$, where g is the conjugation from Proposition 1. Since μ is f_1 and f_2 invariant, $g^*\mu$ is T_p and T_q invariant.

PROPOSITION 7. *$g^*\mu$ is Lebesgue measure λ.*

Proof. By Proposition 5 we know that (S^1, f_1, μ) has positive entropy. Thus $(S^1, T_p, g^*\mu)$ also has positive entropy. Using the result from [J], we know the only measures that are invariant and ergodic for T_p and T_q are λ and measures of entropy zero. □

Proof of main theorem. Rewriting this we get $(g^*)^{-1}\lambda = \mu$. By construction μ is absolutely continuous with respect to Lebesgue measure. We have $(g^*)^{-1}\lambda(A) = \int_A h(x)\,dx$. Now let $A = [0, \tau]$, then $(g^*)^{-1}\lambda(A) = \lambda(gA) = \lambda[0, g(\tau)] = g(\tau)$, because g is an increasing function. So we have $g(\tau) = \int_0^\tau h(x)\,dx$. By the fundamental theorem of calculus $dg(\tau) = h(\tau)$ thus since $h \in \mathscr{C}'$, we have $g \in \mathscr{C}$. □

6. *If there is no common fixed point*

In the construction of the conjugation g in Proposition 1, we used the existence of a common fixed point for f_1 and f_2. Now we want to discuss the situation when no such common fixed point exists, eventually classifying the possible conjugations.

THEOREM 8. *Let f_1 and f_2 be two commuting, expanding, orientation-preserving maps in $\mathscr{C}$ such that f_1 is p-to-1 and f_2 is q-to-1 where p and q generate a nonlacunary semigroup. Then there exists a diffeomorphism $g \in \mathscr{C}$ such that $gf_1g^{-1} = T_p$ and $gf_2g^{-1} = T_q + i/(p-1)$ for some i.*

Proof. Since f_2 is q-to-1, it must have $q-1$ fixed points. By the commutivity of f_1 and f_2, f_1 can only permute these. Thus there is some power k of f_1 that has a common fixed point with f_2 and we can apply our theorem to find a conjugation $\varphi \in \mathscr{C}$ with

$$\varphi^{-1}f_2\varphi = T_q \quad \text{and} \quad \varphi^{-1}f_1^k\varphi = T_{p^k}. \tag{1}$$

Let $\bar{f}_1 = \varphi^{-1}f_1\varphi$. Then $\bar{f}_1$ is a p-to-1 map, T_q is a q-to-1 map, and we can repeat the above process to find an integer n such that $\bar{f}_1$ and T_{q^n} have a common fixed point. Thus there is a conjugation $\psi \in \mathscr{C}$ with

$$\psi^{-1}\bar{f}_1\psi = T_p \quad \text{and} \quad \psi^{-1}T_{q^n}\psi = T_{q^n}. \tag{2}$$

Define a measure $\psi^*\lambda$ by $\psi^*\lambda(A) = \lambda(\psi^{-1}A)$, where λ is Lebesgue measure. Because ψ is smooth we can use an argument much like that in Proposition 6 to show $\psi^*\lambda = \lambda$. But the only continuous, orientation preserving, 1-1 maps that leave Lebesgue measure invariant are rotations. So we will write ψ as R_a.

Using (1) and (2) we have

$$R_a^{-1}\varphi^{-1}f_1\varphi R_a = T_p \quad \text{and} \quad R_a^{-1}\varphi^{-1}f_2\varphi R_a = R_a^{-1}T_qR_a.$$

T_p and $R_a^{-1}T_qR_a$ must commute since f_1 and f_2 do. But $T_pR_a^{-1}T_qR_a(x) = T_p(qx + qa - a) = pqx + pqa - pa$, and $R_a^{-1}T_qR_aT_p(x) = R_a^{-1}T_qR_a(px) = qpx + qa - a$. Thus we get $pqa - pa = qa - a$, or $p(qa - a) = qa - a$ which says $qa - a$ is a fixed point of T_p. That means $qa - a = i/(p-1)$ for some $i = 0, \ldots, p-2$ and thus $a = i/(p-1)(q-1)$.

We can then rewrite $R_a^{-1}T_qR_a(x) = qx + qa - a = qx + a(q-1) = qx + i(p-1)$. Putting $g = R_a^{-1}\varphi^{-1}$ gives the result of the theorem. □

Proposition 9. *If $(p-1)$ and $(q-1)$ are relatively prime then f_1 and f_2 always have a common fixed point. So Theorem 8 in this particular case reduces to the statement of the main theorem.*

Proof. In the proof of Theorem 8 we showed $R_a^{-1}\varphi^{-1}f_1\varphi R_a = T_p$ and $R_a^{-1}\varphi^{-1}f_2\varphi R_a = R_a^{-1}T_qR_a$. So f_1 and f_2 have a common fixed point iff T_p and $R_a^{-1}T_qR_a$ do. If $p-1$ and $q-1$ are relatively prime we can rewrite $a=i/(p-1)(q-1)$ as $i_1/(p-1)+i_2/(q-1)$, and R_a as $R_{a_2}R_{a_1}$. But then we have f_1 conjugate to $R_{a_1}T_pR_{a_1}^{-1}$ and f_2 conjugate to $R_{a_2}^{-1}T_qR_{a_2}$, both conjugations by φR_{a_2}. Notice that both these maps take 0 to 0.

Thus f_1 and f_2 have a common fixed point and we can assume that the conjugation in Theorem 8 sends it to 0. Then Theorem 8 says $gf_2g^{-1}(0)=(T_q+i/(p-1))(0)=0$ which shows that gf_2g^{-1} in fact must equal T_q. Thus Theorem 8 reduces to the statement of the main theorem in this special case. □

We have

$$a\in G=\left\{\frac{l}{(p-1)(q-1)} \bmod 1: l\in\mathbb{Z}\right\}.$$

If $(p-1)$ and $(q-1)$ are relatively prime then this group is the same as

$$H=\left\{\frac{i}{p-1}+\frac{j}{q-1} \bmod 1: i,j\in\mathbb{Z}\right\}$$

and the pair f_1 and f_2 is only conjugate to T_p and T_q. In general these two groups are not the same and we get the following:

Theorem 10. *Let f_1 and f_2 be two commuting, expanding, orientation-preserving maps in $\mathscr{C}$ with f_1 p-to-1 and f_2 q-to-1 where p and q generate a nonlacunary semigroup. Then there exists $g\in\mathscr{C}$ that conjugates the pair to T_p and $R_a^{-1}T_qR_a$ where $a\in G$ is unique up to its coset aH. Thus the number of possible classes for pairs f_1, f_2 is $|G|/|H|$.*

Proof. We need to first show that if f_1, f_2 is conjugate to T_p, $R_{a_1}^{-1}T_qR_{a_1}$ then they are also conjugate to T_p, $R_{a_2}^{-1}T_qR_{a_2}$ where a_2 is an arbitrary element in a_1H. So we will show T_p, $R_{a_1}^{-1}T_qR_{a_1}$ is conjugate to T_p, $R_{a_2}^{-1}T_qR_{a_2}$. Let $a_1=i_1/(p-1)(q-1)$ and $a_2=i_2/(p-1)(q-1)$. Then i_1 and i_2 are such that

$$\frac{i_1}{(p-1)(q-1)}=\frac{i_2}{(p-1)(q-1)}+\frac{c_1}{p-1}+\frac{c_2}{q-1}$$

for some c_1 and c_2 and we can write

$$R_{a_1}=R_{a_2}R_{c_2/(q-1)}R_{c_1/(p-1)}.$$

Using $R_{c_1/(p-1)}^{-1}$ as the conjugation we have T_p, $R_{a_1}^{-1}T_pR_{a_1}$ conjugate to

$$R_{c_1/(p-1)}T_pR_{c_1/(p-1)}^{-1},\quad R_{c_2/(q-1)}^{-1}R_{a_2}^{-1}T_qR_{a_2}R_{c_2/(q-1)}.$$

But the first is equal to T_p and the second to $R_{a_2}^{-1}T_qR_{a_2}$. This gives the result.

Next we need to show that if f_1, f_2 is conjugate to T_p, $R_{a_1}^{-1}T_pR_{a_1}$ and to T_p, $R_{a_2}^{-1}T_qR_{a_2}$ then a_1 and a_2 must be in the same coset. So here we have ψ such that

$\psi^{-1}T_p\psi = T_p$ and $\psi^{-1}R_{a_2}^{-1}T_qR_{a_2}\psi = R_{a_1}^{-1}T_qR_{a_1}$. From the first relationship we see that ψ must have the form $R_{c/(p-1)}$. From the second equation we see that

$$\frac{i_2}{(p-1)(q-1)} + \frac{c}{p-1} = \frac{i_1}{(p-1)(q-1)},$$

which gives the result. □

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