Section 2. Taking Notes

The lecture is a system whereby the notes of the professor become the notes of the student, without passing through the minds of either.

Old saying

A university professor wanted to attend a conference for a few days, but he didn’t want his students to miss out on his wonderful lectures. So he called in his graduate assistant and explained: I’ve taped my lectures for the next several days. All you have to do is bring my tape player to the lecture hall just before class, set it up at the front, and start it. After class, pick it up and insert the next tape. The graduate student agreed and the professor went off happily to the conference.

It happened it wasn’t such a good conference, and he came home a day early. He thought of giving his lecture, but decided instead he would just drop by the lecture hall in the middle of class and see how his system was working. The door was at the front, so as he walked up, all he could see through the glass was the front table. Sure enough, his tape player was going, and he could hear his beautiful lecture.

The professor was about to walk away when he thought: no, let’s see how the students are taking it. So he came right up to the door and angled his head to see back into the room. There were no students! Instead, at every seat, a tape machine was recording his lecture.

Not quite so old story

It doesn’t have to be this bad. First, professors don’t usually give mindless lectures. Second, whatever the professor does, you can make class time valuable by taking notes effectively.

There isn’t one right way to take notes. What works best depends on both your instructor and you. Does your instructor take almost all class examples from the text or make up new ones? (If the former, you can mostly listen and not take notes; if the latter you should write more down.) Do you have a good memory for what happens in class or a poor one? (If the former, your notes can be brief reminders; if the latter, write more, or expand your notes shortly after class.) How well do you read your own handwriting?

Yes, there isn’t just one right way. But there are lots and lots of wrong ways. Most students take too many notes. A researcher once videotaped a course and photocopied all the students’ notebooks. The finding: most students copied down verbatim everything the professor wrote, including the mistakes. In short, they did the same thing with paper that the students in that story did with tape recorders – they transcribed.

What’s wrong is that no digestion took place. It takes a while to digest new ideas and techniques. If you simply copy everything, you have simply postponed the beginnings of understanding until later, when it is less fresh. In fact, copying the board verbatim requires so much attention to what is

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written that you won’t even hear what the professor is saying, which is often the more informative half.

Notetaking, like chewing, should be the first step of digestion.

The opposite of transcribing is taking no notes. Don’t do this either. You’ll forget more than you think. More important, the very process of deciding what to put in your notes makes you concentrate more on the class, and forces you to begin digestion.

The key idea is to listen and participate in class until you see there has been a certain unit of thought. Then write down just enough to capture the key ideas and to allow you to expand them out later. Then go back to listening and participating until you see the next unit. Participating includes asking questions when something is unclear or you don’t see why it is important.

A. Expand Your Notes

Plan to sit down with your notes and expand them. This is the second step of digestion. The sooner you do this after class the better, and the briefer your class notes can be, since you will remember more if the interval is short. Also, you might expand your notes just before or after you read the text portion for the class.

For example, suppose the class concerned optimization in calculus, and the instructor explained the method and did an example. Your notes might contain a brief summary of the example (see the next subsection for details). Expand that summary to a complete example. Add some more mini-examples for any points that are new or difficult for you. You can write things out, as if you were preparing a paper. Or you can present things orally – pretend you are explaining it to a friend. Basically, try to make the material your own, as in §0.0.A.

B. Detailed Examples

Example 1: Calculus optimization. The instructor shows how to find the rectangle of maximum area, given a fixed amount of fence for the perimeter, say 40 units. She chooses a variable (say, $x$ for the width), draws a picture, finds area as a function of $x$, differentiates, sets the derivative to 0, solves for $x$ in this equation (obtaining $x = 10$), and emphasizes that the problem is not finished because $x = 10$ may give a minimum or only a local maximum. She computes the second derivative, finds that it is always negative, and concludes that an endpoint is not the answer and that the area is indeed maximized when $x = 10$. Thus the best rectangle is in fact a square of side 10. In explaining all this, she fills the blackboard and talks for 5 minutes.

What should you write? It depends on whether this example, or one very much like it, is in your text. It also depends on whether this is your first optimization example in class or your tenth, and

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how well you feel you understand the topic. Since this fence example is a common, straightforward example, let’s assume it is the first example in your text.

What is surprising or new to you in the example? As the instructor works through it, anticipate what she will do next. Those places where you anticipate wrong, or have no idea, are where you need good notes.

Let’s say the formulation of the area function, and the work with the derivative, are routine for you. Then summarize them very briefly, say with a sketch and the briefest of algebra. Don’t forget to identify the problem, also very briefly:

Ex.1: max rect area, \( p = 40. \)

\[
\begin{align*}
\begin{array}{c}
20 - x \\
x
\end{array} \\
A = x(20 - x) \\
A' = 0 \implies x = 10
\end{align*}
\]

But suppose you are surprised that the instructor thinks the problem isn’t done at this point. Listen carefully to her as soon as you are surprised. What did you not anticipate? That an endpoint can be the answer? That \( x = 10 \) might be a minimum? That places where \( f' = 0 \) may only be local optima, and you have to give a special argument to show you have the global optimum? If you identify your missing link, put it in your notes carefully.

Suppose the instructor draws pictures of the three types of behavior that a function can have around a point \( a \) where \( f'(a) = 0 \):

\[
\begin{array}{ccc}
\text{(a)} & \text{(b)} & \text{(c)}
\end{array}
\]

Suppose she argues that, for the function \( A(x) \) of this fence problem, 1) the domain is \( 0 \leq x \leq 40 \), 2) \( A' = 0 \) only at \( x = 10 \), and 3) the graph of \( A(x) \) has to look like one of

\[
\begin{array}{ccc}
\text{(a)} & \text{(b)} & \text{(c)}
\end{array}
\]

Suppose she goes on to show that, \( A''(x) = -2 \) for all \( x \) and thus (a) is the correct picture and the area is maximized at \( x = 10 \).

If all of the previous paragraph is what you didn’t anticipate, then you should try to get it all down – in abbreviated form. Since you can sketch the pictures quickly, you should include them. You just need to think of ways to condense the words.

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For instance, you might write

Where $A' = 0$

If only one crit pt

$A'' = -2$, so (a).

As soon as you get a chance after class, expand out the cryptic parts of your notes. Your first line (before your sketch of a rectangle) might become

Ex. 1: Find rectangle of max area if perimeter must = 40.

You might replace the cryptic algebra next to the rectangle in your notes with

$A(x) = x(20 - x) = 20x - x^2$

$A'(x) = 20 - 2x.$

so $A'(x) = 0 \implies 2x = 20 \text{ or } x = 10.$

Expanding the surprising part of the analysis is crucial. You don’t see why having a unique solution to $A' = 0$ limits the curve to the three shapes (a), (b), (c) the instructor drew? Draw a more wiggly curve and observe that the tangent line is horizontal at each local optimum:

A horizontal tangent means that the slope is 0, that is, $A' = 0$. Thus the fact that $x = 10$ is the only solution to $A' = 0$ is crucial to eliminating curves unlike the three the instructor drew.

Now how did she eliminate (b) and (c)? She computed that $A'' = -2$; why was this decisive? You’ll have to recall that the second derivative is the rate of change of the slope, so the slope of the graph of $A(x)$ is always decreasing if $A''(x) = -2$ for all $x$. Therefore the slope must be positive, then

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0, then negative, resulting in figure (a). Had $A''$ been $+2$ for all $x$, figure (b) would have resulted, and the maximum would have occurred at an endpoint. Then you would have had to compute the values of $A(x)$ at the two endpoints to find the larger value.

The more questions you ask yourself as you expand out your notes, the better. You don’t have to write out your answers elaborately, but figure out some sketch and some remarks that record your thoughts.

Example 2: Taylor Polynomials and Series. The Taylor polynomials of a function $f$ approximate $f$ very closely around a base point. When you go to the limit and include infinitely many terms, the resulting Taylor series usually equals $f$.

Suppose your instructor has developed the theory for Taylor polynomials and series in the previous class, and now turns to examples. He begins today by reminding you of the general formulas, writing

$$n^{th} \text{ degree Taylor polynomial for } f(x) \text{ around } a = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Thus Taylor series for $f(x)$ around $a$

$$= \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Now he turns to the first example, writing

Ex 1: Find Taylor Series for $\ln x$ around $a = 1$.

Solution:

$$f(a) = \ln 1 = 0$$

$$f'(x) = \frac{1}{x} \quad f'(a) = 1$$

$$f''(x) = -\frac{1}{x^2} \quad \frac{f''(a)}{2!} = -\frac{1}{1^2} \cdot \frac{1}{2} = -\frac{1}{2}$$

$$f'''(x) = +\frac{2}{x^3} \quad \frac{f'''(a)}{3!} = \frac{2}{3!} = \frac{1}{3}$$

Thus Tay. Series $= 0 + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$

Here is a case where the professor has been rather terse – what he wrote is already like notes. For instance, he hasn’t even stated the general pattern (that the $k$th term of the expansion of $\ln(1-x)$ is $(-1)^{k+1}(x-1)^k/k$), let alone proved it. Instead he has just written out four terms and indicated by ellipsis that you should see a pattern. What should you write?
Since the general formulas are no doubt in your text, and perhaps in your notes from yesterday, there is no point in writing them again – unless you feel the act of writing them will help you get familiar with them. Even then, it would be best to practice writing them after class, when writing them won’t get in the way of paying attention. So, concentrate on the example. What should you write from it?

If this example is in your text (and it is in most texts), you might just state the problem and jot down “try it later”. Or, you might write down the problem statement, one line of computation (say, the line where \( f'''(x) \) is computed) and the answer. You might try to write the answer with summation notation instead of ellipsis (the summation form is \( \sum_{k=0}^{\infty}(-1)^{k+1}(x-1)^k/k \)). This forces you to do some digestion because you have to grasp the pattern. If you don’t figure out how to express the series with summation notation during class, do it when you expand your notes. Also try to convince yourself that the pattern really does hold for all terms, not just through \( f'''(x) \).

Example 3: A proof. Proofs are tricky for taking notes because they are especially hard to comprehend without devoting almost your full attention to listening. You should concentrate on key ideas, both in listening and in what you write. If the professor is not making the key ideas clear, ask questions to try to bring them out.

Suppose you are in a linear algebra course and the professor proves the following theorem.

Theorem. \( B = \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n\} \) is a basis of \( V \) \( \iff \) every \( v \in V \) is a linear combination of \( B \) in a unique way.

After writing just this on the board, she explains what “unique way” means: she writes

- if \( \bar{v} = c_1 \bar{u}_1 + c_2 \bar{u}_2 + \cdots + c_n \bar{u}_n \), and also \( \bar{v} = d_1 \bar{u}_1 + d_2 \bar{u}_2 + \cdots + d_n \bar{u}_n \), then \( c_1 = d_1 \), \( c_2 = d_2 \), \ldots, \( c_n = d_n \), that is, \( c_i = d_i \) for \( i = 1, 2, \ldots, n \).

She also reminds the class what a basis is. She writes

for \( B \) to be a basis of vector space \( V \),

1. \( B \) must span \( V \), and
2. \( B \) must be independent: if \( \sum a_i \bar{u}_i = 0 \), then all \( a_i = 0 \).

The professor now writes the following proof, saying it as she writes.

Basis \( \implies \) unique linear combination:

For any \( \bar{v} \in V \), show that any two linear expressions for \( \bar{v} \) are the same by subtracting:

\[
\bar{v} = c_1 \bar{u}_1 + c_2 \bar{u}_2 + \cdots + c_n \bar{u}_n, \\
\bar{v} = d_1 \bar{u}_1 + d_2 \bar{u}_2 + \cdots + d_n \bar{u}_n, \\
0 = (c_1-d_1)\bar{u}_1 + (c_2-d_2)\bar{u}_1 + \cdots + (c_n-d_n)\bar{u}_n \quad [\text{subtract}]
\]
Since $B$ is a basis, by definition,
\[ c_1 - d_1 = 0, \quad c_2 - d_2 = 0, \quad \ldots, \quad c_n - d_n = 0, \]
that is,
\[ c_1 = d_1, \quad c_2 = d_2, \quad \ldots, \quad c_n = d_n \]
and the two sums for $\vec{v}$ are the same.

Unique combinations $\implies$ basis:

Clearly $0 = \sum_{i=1}^{n} 0\vec{u}_i$ is one way to write $\vec{0}$ as a linear combination of $B$, so by uniqueness it is the only way. By definition, this makes $B$ independent. $B$ spans because by assumption every $\vec{v} \in V$ is a linear combination of $B$ in exactly one way. QED

Here’s what you might write.

Thm. $B = \{\vec{u}_1, \ldots, \vec{u}_n\}$ basis $V$ (span, ind) $\iff$ $\forall \vec{v} \in V$, lin comb unique.

Pf. basis $\implies$ unique

subtract two lin combs of $\vec{v}$, all coefs 0.

\[ \iff \]

$B$ spans (exactly one lin comb per $\vec{v}$)

ind bec. $\sum 0\vec{u}_i$ only way to get $\vec{0}$.

Let’s analyze these notes. Very little is written – abbreviations and symbols keep it short – but the key steps are all there. You can speed things up even further by not bothering to put bars over the vectors, but be sure you understand the distinction between vectors $v$ and numbers $c$ and $d$.

Such brief notes assume you more or less understood the theorem statement and the proof as it was presented. You’ll know for sure when you try to expand your notes. But if you are sure while in class that some things are not clear, then write some more.

For instance, if the idea of unique representation is unfamiliar, include the definition in your notes. If subtraction seems much harder for vectors than for numbers, then include more details. Finally, if the professor emphasizes something while doing the proof (perhaps remarking that subtracting to get an equation for $\vec{0}$ is a frequent proof step in linear algebra) then make some note of what she emphasizes.

At the very least, if something does not make sense in class, put a question mark by it in your notes. Let us suppose the professor’s last sentence puzzles you – why does being able to write each $\vec{v} \in V$ in exactly one way make $B$ span?

When expanding your notes, see if you can expand the algebra to the level the professor wrote. Also, make up some concrete examples for ideas that still seem strange. Pick a basis for a simple space, say, $(1,1)$ and $(1,-1)$ in $\mathbb{R}^2$, and find the unique linear combination for some third vector,
say (2,3). Pick some set that is not a basis for that space (say \((1,1),(1,-1),(1,2)\), or \((1,1)\)) and check that the representations of other vectors aren’t always unique, or don’t always exist. Probably some homework problem assigned after the lecture deals with this point, so you might look through the problems before trying to make up an example from scratch.

Think especially hard about your question marks. During class, the professor reminded you what basis means, but not what span means. Maybe that’s why you don’t understand that part. Look up the definition of span.

Finally, ask yourself why might it be important that bases provide unique linear combinations. If the professor did not address this question, and you are not sure, ask at the next class.

C. Final Advice on Taking Notes

When your instructor gives an example in class, don’t hesitate to ask whether it is from the text, or very close to the text. You need to know before you start taking notes on the example.

Use abbreviations and symbols freely in taking notes, to save time. The less time you take writing notes, the more time to think about the material. See the table of symbols in the Appendix. Make up your own additional abbreviations if you can remember them.

Is rote copying ever OK? In the days before xerox, copying was OK all semester long in an advanced graduate course on a specialty for which no text was available – the notes of the professor really did need to become the notes of the students. But today, if the professor wants to give you extended material not in the text, ask her to make copies for the class in advance, or to put copies on reserve in the library, or to post the material on your local computer network or the World Wide Web.

Sometimes short segments of material are worth copying verbatim. If your professor gives you a theorem not in your text, and wants you to fully understand it, copy down the theorem statement completely. As discussed elsewhere (e.g., §0.0.A), small changes in wording can make huge differences in meaning, and you need to get it right.

Should you ever copy a proof or a solution completely? If you are really lost, and this proof or example is not in your text, then maybe. You might not be able to figure out what’s going on without the details. On the other hand, if you spend all your time copying you may miss the professor’s general remarks that might have helped you. If you do get lost, go to the text, friends, tutoring, or the instructor to get help right away.

If you really must take detailed notes, make sure the professor gives you enough time. Most important, make sure she doesn’t erase before you have recorded what you want. Raise your hand, and if the professor starts to erase without seeing you, yell out “Professor Smith, please wait!”.
Some professors have the annoying habit of standing in one spot, writing only as far as they can reach, and then immediately erasing to write something else. If you ask them to wait enough times, they may get the idea that they can start at one end of the chalkboard and work all the way to the other end before erasing anything.

A good way to slow a professor down is to ask questions. If the only reason you ask a question is to slow her down, then ask her to wait instead of asking a question – professors need to know your real concerns. But if you really are puzzled, then a question is the right thing: it lets the professor know what is hard, it wakes you up, it gives more time for note-taking, and you may get a helpful answer.

What if you completely understand everything? Is it OK to take no notes? Well, you should do something, for otherwise you may fall asleep and miss something you don’t understand. Try to keep one step ahead of the professor; see if you can anticipate what she will say next. Or, create a harder version of the same problem and solve that. Or take out your text and read the next section. Or (heretical thought!) write a letter to your lover, or your mother. In any event, stay awake and keep one ear on the class, in case the subject changes to something you don’t understand.