A Proof by Ideal Elements

Let \( G(X,Y) \) be a bipartite graph with parts \( X,Y \). By a 2-to-1 complete matching of \( X \) to \( Y \) we mean a subgraph in which every \( x \in X \) has degree 2 and every \( y \in Y \) has degree \( \leq 1 \). For any vertex \( v \) in any graph, \( N(v) = \{ u \mid uv \in E \} \) and \( N(S) = \bigcup_{v \in S} N(v) \).

**Theorem.** A bipartite graph \( G(X,Y) \) has a 2-to-1 complete matching from \( X \) to \( Y \) iff

\[
\forall S \subset X, \quad |N(S)| \geq 2|S|. \tag{1}
\]

Proof: We create another bipartite graph \( \tilde{G}(\tilde{X},Y) \) such that \( G \) has a complete 2-to-1 matching from \( X \) to \( Y \) iff \( \tilde{G} \) has a complete matching from \( \tilde{X} \) to \( Y \). Then we show that Hall’s condition on \( \tilde{G} \) is equivalent to (1) on \( G \).

First we define \( \tilde{G} \): If \( X = \{1,2,\ldots,m\} \), then let

\[
\tilde{X} = X \cup X^*.
\]

\( 1^*, 2^*, \ldots, m^* \) are the *ideal elements*.

Next create edges from \( X^* \) to \( Y \) to duplicate the edges from \( X \). That is, for all \( i \in X \), arrange that \( N(i^*) = N(i) \). It is clear that every complete 2-to-1 matching in \( G \) from \( X \) to \( Y \) now corresponds to a complete matching from \( \tilde{X} \) to \( Y \): for each \( i \in X \) just take either edge and move its \( X \) end to \( i^* \). Similarly, given a complete matching from \( \tilde{X} \) in \( \tilde{G} \), for each edge ending at an \( i^* \), move that end to the corresponding \( i \).

To prove Condition (1), henceforth let \( S \) always denote a subset of \( X \) and \( T \) denote a subset of \( \tilde{X} \). Then for all \( T \) define

\[
\tilde{T} = \bigcup_{i \text{ or } i^* \in T} \{i, i^*\},
\]

\[
S_T = \bigcup_{i \text{ or } i^* \in T} \{i\}.
\]

Note that \( S_T \subset X \), and for all \( T \) there exists \( S \) such that \( \tilde{S} = \tilde{T} \) (namely, \( S = S_T \)). Also note that

\[
\forall T \subset \tilde{X}, \quad N(T) = N(\tilde{T}) \quad \text{and} \quad |\tilde{T}| \geq |T|. \tag{2}
\]

Thus we claim

\[
\forall T \subset \tilde{X}, \quad |N(T)| \geq |T| \quad \iff \quad \forall \tilde{T} \subset \tilde{X}, \quad |N(\tilde{T})| \geq |\tilde{T}|. \tag{3}
\]

Certainly (3)\( \Longrightarrow \) (4) since the \( \tilde{T} \) sets are special cases of \( T \) sets; and (4)\( \Longrightarrow \) (3) by (2): for any \( T \),

\[
|N(T)| = |N(\tilde{T})| \geq |\tilde{T}| \geq |T|. \tag{4}
\]

Finally, to prove (1), we show that it is equivalent to Hall’s condition:

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X has a complete 2-to-1 matching in $G$

\[\iff \tilde{X} \text{ has a complete matching in } \tilde{G}\]

\[\iff \forall T \subset \tilde{X}, \ |N(T)| \geq |T| \quad \text{[Hall’s Thm]}\]

\[\iff \forall \tilde{T} \subset \tilde{X}, \ |N(\tilde{T})| \geq |\tilde{T}|\]

\[\iff \forall S \subset X, \ |N(\tilde{S})| \geq |\tilde{S}| \quad \text{[every } \tilde{T} \text{ is an } \tilde{S}]\]

\[\iff \forall S \subset X, \ |N(S)| \geq 2|S|. \quad \Box \quad [|\tilde{S}| = 2|S|]\]

Query to check your understanding:

Suppose $U = \tilde{T}$ for some $T$. Then for how many $T$ is $U = \tilde{T}$?