Let $M = [a_{ij}]$ be a matrix. The **matrix norm** $\|M\|$ is defined by

$$\|M\| = \max\{|Mx| : |x| = 1\}.$$ 

In short, $\|M\|$ is the farthest from the origin (in the image space) to which $M$ maps a point from the unit sphere (in the domain space). This maximum exists because $|Mx|$ is a continuous real-valued function restricted to $|x| = 1$, a compact domain, and therefore attains a maximum.

$\|M\|$ is different from $\sqrt{\sum_{i,j} a_{ij}^2}$, the norm $M$ would have if you regarded it as a vector (say, by stringing all the entries out in one row). Let us call that second norm $|M|_E$ (for Euclidean norm).

1. Consider the matrix $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Show that $\|M\|$ and $|M|_E$ are different. (For this simple matrix, both norms are easy to compute without any theory.)

2. Show that, for every $x$, $|Mx| \leq \|M\||x|$. Thus $Mx = O(x)$ for every matrix $M$, something we have long asserted, but which we had to waive our hands to prove previously. Now we can apply the definition of $O$ directly.

3. A function $f$ is **uniformly continuous** if

$$\forall \epsilon \exists \delta \forall a \quad |x - a| < \delta \implies |f(x) - f(a)| < \epsilon.$$ 

In other words, you can pick $\delta$ solely in terms of $\epsilon$; you don’t have to pick different $\delta$’s depending on $a$. (Not every function is uniformly continuous, as you can see by considering $f(x) = x^2$; for any $\epsilon$ vertical tolerance, the bigger $a$ is, the steeper the graph is and the narrower the $\delta$ tolerance has to be.)

Prove: any linear transformation $T$ is uniformly continuous. Hint: let $M$ be the matrix of $T$ and use the matrix norm.

Note: the definition of regular continuity (for a function continuous everywhere) is

$$\forall a \forall \epsilon \exists \delta \quad |x - a| < \delta \implies |f(x) - f(a)| < \epsilon.$$ 

Notice the difference in order of quantifiers. That difference really makes a difference!

4. Let $S$ be a $n \times n$ symmetric matrix, such as one gets by taking all the second partial derivatives of some continuously differentiable $f : \mathbb{R}^n \to \mathbb{R}$. Consider the quadratic form $Q(x) = x^T S x$. Then it is a theorem of linear algebra that there is a matrix $\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, and for each $x$ there is another vector $v$, such that

1. $|x| = |v|$, and

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2. $Q(x) = v^T \Lambda v$.

(This is because $S$ can be diagonalized using an orthonormal change of basis; $v$ is just $x$ rewritten relative to this other basis. The $\lambda_i$ in $\Lambda$ are just the eigenvalues of $S$, which are necessarily real numbers. We may take $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.)

a) Show that, for all $x \in \mathbb{R}^n$,

$$\lambda_1 |x|^2 \geq Q(x) \geq \lambda_n |x|^2,$$

with each extreme obtained for some vectors. Hint: rewrite everything in terms of $v$ and $\Lambda$.

b) Let $c = \max\{|\lambda_1|, |\lambda_n|\}$. Show that $\|S\| = c$.

5. In the previous problem, we found a simple formula for $\|M\|$ in terms of eigenvalues, when $M$ is symmetric. This formula can’t possibly work for general matrices $M$, because $M$ might not even be square, and thus won’t have any eigenvalues. Even if $M$ is square, the eigenvalues might not be real. Even if the eigenvalues are real, the formula for $\|M\|$ in the last problem doesn’t always hold. Let $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

a) Show that all the eigenvalues are 1.

b) Find a unit vector $x$, such that $|Mx| > 1$, thus showing $\|M\| > 1$.

6. The discussions in Edwards, Sect II.4 and II.8, show how to find $\|M\|$ for any matrix using Lagrange multipliers. However, there is a much cleaner way using linear algebra. See if you can prove: for any $M$,

$$\|M\| = \sqrt{\mu}, \quad \text{where } \mu \text{ is the largest eigenvalue of } M^T M.$$

Hint: rewrite $|Mx|^2 = (Mx) \cdot (Mx)$ using matrix multiplication.

7. Return to the matrix $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

a) Find a unit vector $u$ such that $|Mu|$ is maximum (for unit vectors).

b) Find a unit vector $v$ such that $|Mv|$ is minimum. Is there any nice geometric relationship between $u$ and $v$?

c) The set of points $\{x = (x, y) : |Mx| = 1\}$ is an ellipse. Can you find a quadratic equation in $x$ and $y$ for this ellipse? (It won’t be as simple as $ax^2 + by^2 = 1$ because the axes of the ellipse aren’t the $x$- and $y$-axes, but there must be a equation of the form $ax^2 + bxy + cy^2 = 1$.

d) The set $\{Mx : |x| = 1\}$ is an ellipse. Can you find a quadratic equation for this ellipse? What do the lengths of the axes of this ellipse have to do with $\|M\|$?
8. Let \( \mathbf{a} \) be a critical point of \( f : \mathbb{R}^n \to \mathbb{R} \), so that
\[
f(\mathbf{a} + \mathbf{h}) = f(\mathbf{a}) + \mathbf{h}^T S \mathbf{h} + R(\mathbf{h}),
\]
where \( S \) is the symmetric matrix of second partials (the “Hessian”), and \( R(\mathbf{h}) \in O(|\mathbf{h}|^3) \). Suppose \( S \) is positive definite with minimum eigenvalue \( \lambda_m \). Let \( C \) be the constant such that \( R(\mathbf{h}) \leq C|\mathbf{h}|^3 \) for \( \mathbf{h} \) sufficiently small. In terms of these quantities, find a \( \delta \) such that
\[
0 < |\mathbf{h}| < \delta \implies f(\mathbf{a} + \mathbf{h}) > f(\mathbf{a}).
\]
In other words, prove conclusively that if the Hessian is positive definite at a critical point \( \mathbf{a} \), then \( \mathbf{a} \) is a local minimum of \( f \) (and there are no other local minima within \( \delta \) of \( \mathbf{a} \)).

9. Determine if the quadratic form
\[
Q(x, y, z) = x^2 + 3y^2 + 5z^2 + 4xy + 6xz + 8yz
\]
is positive definite, negative definite, or neither. Hint: write \( Q \) in the form \( \mathbf{x}^T A \mathbf{x} \), where \( A \) is a symmetric matrix. The principal determinant method is easy to apply, the eigenvalue method is more tedious, at least if you do it without machine help.