More Contour Integration and Exact Differentials

1. I stated in class that if $\omega = df$, then line integrals of $\omega$ are independent of path. I showed that $\omega = df$ is true if $\omega$ is the 1-form for gravity. But I also showed that if $\omega = \frac{(-y \, dx + x \, dy)}{x^2 + y^2}$, then $\omega = d\theta$, where $\theta = \arctan(y/x)$. Therefore, $\int_C \omega$ should be 0 for this $\omega$, if $C$ is a closed path. Yet I claimed that, for any circle $\mathcal{C}$ around the origin, $\int_{\mathcal{C}} \omega = 2\pi$. Something doesn’t jibe. What’s the explanation?

2. Consider the two circles

$$C_1 : x^2 + y^2 = 1, \quad C_1 : x^2 + (y + 2)^2 = 16.$$ Let $D$ be the region between the circles. In the integrals below, go around each circle counterclockwise.

a) Draw $C_1$, $C_2$ and $D$.

b) Consider $\omega = P \, dx + Q \, dy = x \, dx + y \, dy$. Verify that $Q_x - P_y = 0$ on $D$. Therefore the two integrals

$$\int_{C_1} \omega \quad \text{and} \quad \int_{C_2} \omega \tag{2}$$

must be equal. Verify this by direct calculation. Which integral was easier to compute?

c) Draw a picture of $C_1$ with the vectors of $F = (P, Q)$ attached. That is, at point $\mathbf{x}$, attach $F(\mathbf{x})$ with its tail at $\mathbf{x}$. Explain why it is now obvious that $\int_{C_1} \omega = 0$. However, it is not obvious (to me) from the same sort of picture that $\int_{C_2} \omega = 0$.

d) For this $\omega$, $Q_x - P_y = 0$ inside $C_1$ as well as in $D$. Why does this ensure that both integrals in (2) are 0?

e) Now consider

$$\alpha = \frac{-y \, dx + x \, dy}{x^2 + y^2}.$$ Verify that $d\alpha = 0$ in $D$. Therefore

$$\int_{C_1} \alpha = \int_{C_2} \alpha. \tag{3}$$

Verify this by direct computation, or at least try. Which integral is easier to compute? (This part of the problem is supposed to amaze you – that two integrals that look so different and are not related by the Chain Rule must be equal.) Note that $d\alpha$ isn’t 0 everywhere inside of $C_1$, because $d\alpha$ is not defined everywhere inside $C_1$. Where isn’t it defined? Thus no theorem ensures that $\int_{C_1} \alpha = 0$. And sure enough, $\int_{C_1} \alpha \neq 0$.

3. Consider

$$\omega = e^{(x+2y)^2} \, dx + 2e^{(x+2y)^2} \, dy.$$ Show that $\omega = df$ for some $f$. (Whereas for some of the similar problems in Edwards you can actually integrate to find $f$ instead of using the necessary derivative condition, here you cannot integrate, at least not in closed form.)

April 25, 1997