1. Definitions.

The symbol $\ni$ means “such that”; another shorthand is s.t.

The open ball of radius $r > 0$ around $a$, denoted $B_r(a)$, is defined to be $\{x : |x - a| < r\}$.

The closed ball of radius $r \geq 0$ around $a$, denoted $\overline{B}_r(a)$, is defined to be $\{x : |x - a| \leq r\}$.

The sphere of radius $r \geq 0$ around $a$, denoted $S_r(a)$, is defined to be $\{x : |x - a| = r\}$.

$S$ is a neighborhood of $a$ if $\exists \epsilon \ni B_\epsilon(a) \subset S$.

The collection of all neighborhoods of $a$ is denoted $\mathcal{N}_a$.

$S$ is an open set if $\forall x \in S$, $\exists \epsilon \ni B_\epsilon(x) \subset S$. In other words, $S$ is open $\iff S$ is a neighborhood of each of its points.

The collection of all open sets (in the domain space or range space of a function) is denoted $\mathcal{O}$.

Thus we may write: $S \in \mathcal{O} \iff \forall a \in S$, $S \in \mathcal{N}_a$.

The image of set $S$ under function $f$, denoted $f(S)$, is $\{f(x) \mid x \in S\}$.

The preimage of set $T$ under function $f$, denoted $f^{-1}(T)$, is $\{x \mid f(x) \in T\}$.

$f$ is neighborhood continuous at $a$ means

$$\forall T \in \mathcal{N}_{f(a)}, \ f^{-1}(T) \in \mathcal{N}_a.$$  

$f$ is open-set continuous means

$$\forall T \in \mathcal{O}, \ f^{-1}(T) \in \mathcal{O}.$$  

2. Theorem. $f$ is $(\epsilon-\delta)$ continuous at $a$ $\iff f$ is neighborhood continuous at $a$.

3. Theorem. $f : R^n \rightarrow R^m$ is $(\epsilon-\delta)$ continuous everywhere in its domain $\iff f$ is open-set continuous.

Note: if the domain $D$ of $f$ is an open set in $R^n$ instead of $R^n$, the same theorem holds. However, if $D \notin \mathcal{O}$, then the theorem has to be modified to talk about relative open sets. This concept will be discussed at a later time – perhaps in a later course.
4. To prove the Theorems above (not shown here), the following equivalences are helpful. Each line below is equivalent to each other line.

\[ |x - a| < d \implies |f(x) - f(a)| < \epsilon \]

\[ x \in B_\delta(a) \implies f(x) \in B_\epsilon(f(a)) \]

\[ B_\delta(a) \subset \{x \mid f(x) \in B_\epsilon(f(a))\} \]

\[ B_\delta(a) \subset f^{-1}(B_\epsilon(f(a))) \]

\[ \{f(x) \mid x \in B_\delta(a)\} \subset B_\epsilon(f(a)) \]

\[ f(B_\delta(a)) \subset B_\epsilon(f(a)) \]