\[(\hat{A} \implies \hat{B}) \implies (A \implies B)\]

In class we proved that the Inverse Function Theorem implies the Implicit Function Theorem. Both theorems are of the form \( A \implies B \). Therefore, what we proved had the logical form of the title above (where the hat implication is the Inverse Function Theorem). Many claims in mathematics are of this form, and the proof we did is a very good example of how such proofs tend to work. Here’s what we did.

1. Took any \( g \) that satisfies \( A \).
2. Constructed from it a \( \hat{g} \) that satisfies \( \hat{A} \).
3. Therefore, by the Inverse Function Theorem, \( \hat{g} \) satisfies \( \hat{B} \).
4. Deconstructed \( \hat{g} \) to show that \( g \) satisfies \( B \).

Above, \( A \) is the hypothesis of the Inverse Function Theorem: \( g(x, y) \) is continuously differentiable and \( \partial g / \partial y \) is invertible at \((a, b)\). \( B \) is the conclusion: there exists a continuously differentiable \( h \) such that \( h(a) = b \) and around \( a \), \( g(x, h(x)) = 0 \). Similarly, \( \hat{A} \) and \( \hat{B} \) are the hypothesis and conclusion of the Inverse Function Theorem.

In our example, and more generally, to prove

\[(\hat{A} \implies \hat{B}) \implies (A \implies B)\]

the best place to start is usually

Assume \( A \).

Only later do you bring in \( \hat{A} \implies \hat{B} \). The final conclusion is \( B \).

I have a picture in my head for this sort of proof (see below). \( A \) and \( B \) are on opposite sides of a river and we have to get across. A known implication is a bridge. There isn’t a bridge leaving directly from \( A \), but there is one from \( \hat{A} \) to \( \hat{B} \). So we walk upstream. Walking upstream from \( A \) to \( \hat{A} \) amounts to constructing \( \hat{g} \) from \( g \). Walking downstream on the other side amounts to deconstructing \( \hat{g} \) to show that \( g \) has the desired properties.