1. Artificial potential field navigation

Consider a Turtlebot with state \( q = (x, y, \theta) \). We would like to steer it through the world using an artificial potential field approach.

a. The potential associated with some point obstacle \( i \) located at \((x_i, y_i)\) is defined as a Gaussian shape:

\[
U_i(x, y) = \alpha e^{-\frac{(x-x_i)^2 + (y-y_i)^2}{2\sigma^2}}
\]

for some scaling factor \( \alpha \) and falloff rate \( \sigma \). What is the force \( F_i = \nabla U_i \) acting on the robot due to this potential? That is, what is the gradient of \( U_i \) with respect to \( x \) and \( y \)? Explain why the force vanishes quickly as the robot gets further away from the obstacle.

b. Propose a control law to set the robot’s desired linear velocity \( \dot{x}_R \) and angular velocity \( \dot{\theta}_R \), based on its current state \( q \) and an artificial potential force \( F = (F_x, F_y) \). The controller you design should have the effect of steering the robot to a minimum of the artificial potential field as it drives along.

2. Linearity of expectations

The expected value of some function \( h(x, y) \) of two variables with known joint probability distribution \( p(x, y) \) is defined (in the discrete case) as

\[
E[h(x, y)] = \sum_x \sum_y h(x, y) p(x, y)
\]

Prove that expectations are linear. If \( h(x, y) = af(x) + bg(y) \) is the linear combination of two functions \( f \) and \( g \) that depend only on \( x \) and \( y \) respectively, show that

\[
E[af(x) + bg(y)] = a \left( \sum_x f(x) p(x) \right) + b \left( \sum_y g(y) p(y) \right)
\]

for any real numbers \( a \) and \( b \). \textit{Hint: you can’t use linearity of expectations here: that’s what you’re proving! Instead, try to use the definition of conditional probability along with the law of total probability.}
3. Covariance of a random variable and a derived quantity

Let $x, y \in \mathbb{R}$ be random variables with known means $\mu_x, \mu_y$, variances $\text{Var}(x), \text{Var}(y)$, and covariance $\text{Cov}(x, y)$. Define $z \in \mathbb{R}$ as the random variable resulting from a linear combination of $x$ and $y$:

$$z = ax + by$$

where $a, b \in \mathbb{R}$ are known constants.

**a.** Prove that

$$\text{Cov}(x, z) = a \text{Var}(x) + b \text{Cov}(x, y)$$

by using linearity of expectations, and the definition of variance and covariance.

**b.** Explain why this is relevant to the derivation of the Kalman Filter from Tuesday’s class. What are the Kalman filter analogues to $a, b, x,$ and $y$?