Physics 138
Plasma Physics 2003
Seminar 6 presentation (mb)

Plasma resistivity (mb):

Goldston: The simple picture of the drag force on electrons starts with

$$\mathbf{F} = -n_e m \left\langle \nu_{ei} \mathbf{v} \right\rangle.$$

This is nothing more than the drag force density ($=$ number density x mass x acceleration). We should use the small angle ν_{ei} that has the extra factor of $4 ln\Lambda$ which emphasizes the idea that the cumulative effect of small deflections is more important than the occasional close encounter (see notes from seminar 1). The angled brackets $\langle \rangle$ mean that we should average over all the particles. Electrons with different v's have different ν_{ei}'s. To do the averaging properly, we need to integrate over the correct (self-consistent) electron distribution function. We can approximate this with a drifting Maxwellian (the drifting is what generates the current).

We start by expanding the drifting Maxwellian $exp(-|\mathbf{v} - \mathbf{u}|^2/2v_t^2) = exp(-|v^2 - 2\mathbf{u} \cdot \mathbf{v} + u^2|/2v_t^2) \cong exp(-|v^2|/2v_t^2)(1 + \mathbf{u} \cdot \mathbf{v})$ where we assume that the drift velocity is small ($u/v_t \ll 1$). We threw out the term $(u/v_t)^2$ entirely and we use the Taylor expansion $exp(\delta) \cong 1 + \delta$ for the cross term. This gives us the expression in the middle of p. 171.

Now the integral we want to do to get the z-directed drag force (from the first line above) is:

$$F_z = -m \int f_e(v)\nu_{ei} v_z d^3v.$$

Recalling the form of ν_{ei} for a single electron of velocity v:

$$\nu_{ei} = \frac{n_i Z^2 e^4 ln\lambda}{4\pi \epsilon_0^2 m^2 v^3}$$

where the v^{-3} dependence is evident and must be integrated over. If we substitute our expanded form of f_e, we find that the leading term vanishes so we have:

$$F_z = -m u_z \int f_e(v)\nu_{ei} \frac{v_z^2}{v_t^2} d^3v$$

The overall scaling of the integrand is like $exp(-|v^2|/2v_t^2)/v$ (we'll see this later).

The final result is $F_z = -n_e m \left\langle \nu_{ei} \right\rangle u_z$ where now the distribution function averaged collision frequency is given by:

$$\langle \nu_{ei} \rangle = \frac{2^{1/2} n_i Z^2 e^4 ln\lambda}{12\pi^{3/2} \epsilon_0^2 m^{1/2} T^{3/2}}.$$

Notice that the functional form is the same as a mono-energetic beam but now this form is averaged over a drifting Maxwellian.

This derivation isn't quite right since the real distribution function for electrons flowing due to an electric field isn't as simple as a drifting Maxwellian. Goldston discusses this at the top of p. 176. The idea is that high velocity electrons respond differently than low velocity ones. The upshot of this more accurate theory (called Fokker-Planck) is a factor of 2 downstairs (ie $2^{1/2}$ appears in the denominator).

The loss of electron momentum is contained in the term:

$$\mathbf{R_{ei}} = -mn_e \langle \nu_{ei} \rangle (\mathbf{u_e} - \mathbf{u_i}).$$

Using $\mathbf{E} = \eta \mathbf{J}$ (with $\mathbf{J} = -n_e e(\mathbf{u_e} - \mathbf{u_i})$) and $\mathbf{R_{ei}} = en_e \mathbf{E}$ and the result above we get:

$$\eta = \frac{m \langle \nu_{ei} \rangle}{n_e e^2} = \frac{m^{1/2} Z e^2 ln\lambda}{2^{1/2} 12\pi^{3/2} \epsilon_0^2 T^{3/2}}.$$

Note that there is no explicit density dependence (except for the logarithmic dependence in the cutoff factor). A useful form the plasma resistivity is

$$\eta = 5.15 \times 10^{-5} \frac{Z ln\lambda}{T_e^{3/2}} \Omega - m.$$

For comparison, the resistivity of copper is $1.72 \times 10^{-8}\Omega - m$, aluminum is 2.63, mercury is 94, stainless steel is about 140. So we see that plasma at $T_e = 960$ eV has a resistivity like copper, and at $T_e = 50$ eV has a resistivity like stainless steel (SSX is a bit more resistive that stainless steel).

Fokker-Planck: We're going to skip the derivation of the Fokker-Planck equation for now (maybe towards the end of the semester, someone would like to do this as a project). In any case, the Fokker-Planck equation governs how the distribution function a group of test particles (say electrons carrying current) is (1) slowed down due to friction with background particles (called by Chandrasekhar "dynamical friction") and (2) spread out due to velocity space diffusion.

I'll write the form of the Fokker Planck equation on the board for reference. The key point is that the form of the dynamical friction term scales like $erf(\xi)/\xi$ where $\xi = u/u_{th}$. The function $erf(\xi)/\xi$ has a maximum at about $\xi = 0.9$ (peak value is $erf(\xi)/\xi = 0.43$) which means that dynamical friction becomes less and less for velocities above the thermal speed.

Dreicer field and "runaway electrons": If the electron temperature is high and the density is low, we find that there is no equilibrium between

the acceleration due to an applied electric field driving some current and the drag due to dynamical friction. Another way to put it is if the electric field is above some critical value, then the dynamical friction lies to the right of the maximum (above $\xi = 0.9$) and we have "runaway electrons". Notice that the "self-consistent" electron distribution function is decidedly non-Maxwellian in this case. It has a long "tail" out above the thermal speed. This also means that the notion of a collision controlled steady state drift velocity for charged particles is of limited applicability. The form of this critical field (called the Dreicer field after it's discoverer) is:

$$E_{Dr} = 0.43 \frac{n_e Z e^2 ln\lambda}{8\pi\epsilon_0^2 T_e} = 5.6 \times 10^{-18} n_e Z \frac{ln\Lambda}{T_e} \; V/m.$$

In SSX, we get runaways for electric fields greater than about $E_{Dr} \cong 300V/m = 3V/cm$ (for $n_e = 10^{20}m^{-3}$ and $T_e = 20eV$). Our peak reconnection electric fields are $E_{rec} \cong vB = (10^4 m/s)(0.1T) = 1000V/m$ so we likely have a "tail" of runaways.

References: Goldston, Bellan chapter 11, Spitzer "*Physics of fully ionized gases*", R.S. Cohen, L. Spitzer, Jr., and P. McRoutly, "The electrical conductivity of an ionized gas", Phys. Rev. **80**, 230 (1950), L. Spitzer, Jr. and R. Harm, "Transport phenomena in a completely ionized gas", Phys. Rev. **89**, 977 (1953), H. Dreicer, "Electron and ion runaway in a fully ionized gas", Phys. Rev. **115**, 238 (1959).