Plasma gun notes

Here are some notes based on an idea of Paul Bellan’s (see his Spheromak book for more details). It’s a nice demonstration of Hamilton’s equations of motion. It also shows how magnetic flux can be a canonical momentum variable... the electrodynamics comes for free. The equations of motion are not analytically soluble (I think) but shouldn’t be hard to solve numerically. Give it a try (with Mathematica or IDL)!

1. Hamilton’s equations:

   The Hamiltonian is formally:
   \[ H = \sum_j p_j \dot{q}_j - L = T + U \]

   where \( L \) is the Lagrangian, \( p_j \) are the canonical momenta, and \( q_j \) are the canonical coordinates. The coordinates and momenta are connected through the Lagrangian:
   \[ p_j = \frac{\partial L}{\partial \dot{q}_j} \]

   Hamilton’s equations are
   \[ \frac{\partial H}{\partial \dot{q}_j} = -\dot{p}_j \quad \frac{\partial H}{\partial p_j} = \dot{q}_j \]

   The Hamilton approach is elegant and exposes conservation laws.

2. Plasma gun:

   We can model the plasma gun as coaxial inductor with inductance \( L(x) = L_0 + Lx \) and a fixed capacitor on the back end \( C \). \( L \) is an inductance per unit length and \( L_0 \) is the inductance of the system before the spheromak starts to move. The spheromak is a sliding short of mass \( m \) impaled on the center electrode. The potential energy of the system (before current starts to flow and mass begins to move) is:
   \[ U = \frac{q^2}{2C} \]

   The “kinetic” energy of the system is:
   \[ T = \frac{p^2}{2m} + \frac{1}{2} L(x)\dot{q}^2 \]
The Lagrangian of the system is then:

\[ L = T - U = \frac{p^2}{2m} + \frac{1}{2} L(x)q^2 - \frac{q^2}{2C} \]

We can get the appropriate canonical momenta from:

\[ p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \text{and} \quad p_q = \frac{\partial L}{\partial \dot{q}} = L(x)\dot{q} \]

Note that the general momentum \( p_q \) isn’t a momentum at all! It’s the magnetic flux behind the spheromak (\( \Psi \equiv L(x)\dot{q} \)).

You can write the Hamiltonian for the system in terms of the canonical variables:

\[ H(x, p_x, q, p_q) = \sum_j p_j \dot{q}_j - L = \frac{p_x^2}{2m} + \frac{p_q^2}{2L(x)} + \frac{q^2}{2C} = \frac{p^2}{2m} + \frac{1}{2} L(x)\dot{q}^2 + \frac{q^2}{2C} \]

There are 4 Hamilton’s equations using the 2 canonical coordinates \((x, q)\) and the associated canonical momenta \((p, \Psi \equiv L(x)\dot{q})\):

\[ \frac{\partial H}{\partial x} = -\frac{p_q^2}{2L^2(x)} = -\frac{1}{2} \mathcal{L}\dot{q}^2 = -\dot{p} = -m\ddot{x} \quad \text{(main eq of motion)} \]

\[ \frac{\partial H}{\partial p_x} = \frac{p_x}{m} = \dot{x} \quad \text{(def of linear mom)} \]

\[ \frac{\partial H}{\partial q} = \frac{q}{C} = -\dot{p}_q = -\dot{\Psi} \quad \text{(Faraday, def of mom canonical to q)} \]

\[ \frac{\partial H}{\partial p_q} = \frac{p_q}{L(x)} = \frac{L(x)\dot{q}}{L(x)} = \dot{q} \quad \text{(tautology)} \]

The initial conditions are the spheromak at \( x = 0 \) and the capacitor fully charged. As the capacitor drains, current flows and so a force is applied to the spheromak.