Kolmogorov scaling (different indices)

The essence of the Kolmogorov 1941 scaling argument for the omnidirectional wavenumber spectrum for fully developed turbulence is that $E(k)$ depends only on $k$ (via a power-law) and also on the energy transfer rate $\epsilon$. Kolmogorov thought about an energy rate per unit mass: $\epsilon \sim v^2/\tau$. For us, we think about magnetic energy so $\epsilon \sim b^2/\tau$, where $b$ is the fluctuating part of the magnetic field, and $\tau$ is the time scale over which the energy is transferred.

The dimensions of $E(k)$ are such that:

$$\int E(k)\,dk = \langle b^2 \rangle$$

so $E(k) \propto b^2/k$. The time $\tau$ in the energy transfer rate depends on the physics of the transfer. For MHD, we consider an Alfvén crossing time at the scale $L$:

$$\tau_{\text{MHD}} = \frac{L}{v_A} \sim \frac{1}{kb}.$$  

This is because $\omega_{\text{MHD}} = kv_A$. So now we do dimensional analysis:

$$E(k, \epsilon) = Ck^{\alpha} \epsilon^\beta$$

$$\frac{b^2}{k} = Ck^{\alpha} \left( \frac{b^2}{\tau_{\text{MHD}}} \right)^\beta = Ck^{\alpha} b^{2\beta} (kb)^\beta$$

We find that $2 = 3\beta$ or $\beta = 2/3$ and $-1 = \alpha + \beta$ so $\alpha = -5/3$. We get the famous Kolmogorov 1941 result:

$$E(k) = Ck^{-5/3} \epsilon^{2/3}$$

An interesting twist happens if the time scale for the transfer is faster, say due to Whistler waves or kinetic Alfvén waves. In that case, there’s a different dispersion relation (see below). We get that $\omega_{\text{Hall}} = k^2 \delta_i v_A = k^2 \delta_i^2 \omega_{ce}$, or essentially:

$$\tau_{\text{Hall}} \sim \frac{1}{k^2 \delta_i}.$$  

That extra factor of $k$ changes the scaling for $E(k)$ at scales smaller than $\delta_i$.

$$E(k, \epsilon) = Ck^{\alpha} \epsilon^\beta$$
\[
\frac{b^2}{k} = Ck^\alpha \left( \frac{b^2}{\tau_{\text{Hall}}} \right)^\beta = Ck^\alpha b^2 \beta (k^2 b) \beta
\]

We find that \( 2 = 3\beta \) or \( \beta = 2/3 \) and \(-1 = \alpha + 2\beta \) so \( \alpha = -7/3 \). We get a modified energy spectrum:

\[
E_{\text{Hall}}(k) = C k^{-7/3} \epsilon^{2/3}
\]

**Dispersion relations:** The dispersion relation for Whistler waves comes from the dispersion relation for R-waves (see Bellan, or any plasma book):

\[
\frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega^2_{pe}/\omega^2}{1 - \omega_{ce}/\omega}.
\]

For SSX, the frequencies are always low compared to electron physics so \( \omega \ll \omega_{pe}, \omega_{ce} \), so

\[
\frac{c^2 k^2}{\omega^2} = 1 + \frac{\omega^2_{pe}}{\omega_{ce} \omega}.
\]

Furthermore, SSX plasmas are always over-dense, meaning \( \omega_{pe}/\omega_{ce} \gg 1 \) (about 100 typically). So the dispersion relation becomes:

\[
\frac{c^2 k^2}{\omega^2} = \frac{\omega^2_{pe}}{\omega_{ce} \omega}
\]

\[
\omega_{\text{Hall}} = \frac{c^2 k^2 \omega_{ce}}{\omega^2_{pe}} = \delta_e \omega_{ce} k^2 = \delta_i v_A k^2
\]

The key point is that the dispersion relation depends on \( k^2 \) (i.e. is dispersive) but it turns out that \( \delta^2_e \omega_{ce} = \delta_i v_A \) (which is also interesting).

On my website, there are some notes called alpha scaling, but the basic story is that:

\[
\alpha \equiv \tau_{\text{Alf}} \omega_{ci} = L \omega_{ci}/v_A = L/\delta_i
\]

this says the number of orbits an ion executes in a characteristic dynamical time (the time it takes an Alfvénic disturbance to move a distance \( L \)) is the same as the number ion inertial lengths in \( L \). Another way to write it is \( v_A = \delta_i \omega_{ci} \). From there, it’s easy to show that \( \delta_i v_A = \delta_e^2 \omega_{ce} \) (i.e. the form I used in the dispersion relation above) by keeping track of factors of \( M/m \).