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Mathematical Explanation in Science

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ABSTRACT

Does mathematics ever play an explanatory role in science? If so then this opens the way for scientific realists to argue for the existence of mathematical entities using inference to the best explanation. Elsewhere I have argued, using a case study involving the prime-numbered life cycles of periodical cicadas, that there are examples of indispensable mathematical explanations of purely physical phenomena. In this paper I respond to objections to this claim that have been made by various philosophers, and I discuss potential future directions of research for each side in the debate over the existence of abstract mathematical objects.

1 Introduction: Mathematical Explanation

The recent rise of philosophical interest in the topic of mathematical explanation can be divided into two main strands. (See Mancosu [2008] for a useful overview of this literature.) One strand has focused on 'internal' mathematical explanation, in other words the role of explanation within mathematics, for example in distinguishing between more and less explanatory proofs of a

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particular theorem. A second strand has focused on 'external' mathematical explanation; in other words, the potential role of mathematics in science as a tool for providing explanations for physical phenomena. It is this second, external sense of mathematical explanation that is my primary focus in this paper.

As I hope to show, there may well be broader philosophical pay-offs, both methodological and ontological, from investigating this topic in more detail. On the methodological side, an examination of the contribution mathematics can make to explanations in science has the potential to cast light on the nature of scientific explanation and on the viability of different models of scientific explanation. I won't have much to say about this side of things here, since I want instead to concentrate on the link between mathematical explanation and ontology. The idea, in a nutshell, is to use inference to the best explanation to draw conclusions concerning the existence of mathematical entities that feature in scientific explanations. This approach is not a new one—as we shall see its roots go back several decades—but assessing its merits depends crucially on the still relatively unexplored topic of mathematical explanation in science.

2 Indispensability and Explanation

There is a well-known, ongoing debate concerning the proper ontology for mathematics, between platonists on one side and nominalists on the other, that arises from the Quine-Putnam indispensability argument. In its basic form, this argument proceeds from the claim that mathematical objects play an indispensable role in our best scientific theories to the conclusion that we therefore ought rationally to believe in the existence of these mathematical objects. In other words, scientific realists ought also to be mathematical platonists. As an arch-holist, Quine himself was relatively uninterested in delineating what precise kinds of roles mathematical objects play in science. All that is important is the eliminability or otherwise of the various posits of a given theory. However, this lack of concern for how mathematics works in science has left the classic Indispensability Argument open to a variety of objections that draw parallels between aspects of applied mathematics and other facets of scientific theorizing, which seem less ontologically serious. Maddy ([1992]) has argued that episodes from the history of science show that scientists do not in general take a holistic attitude to their theories insofar as they do not treat all posits as being on a par. Maddy also points to idealized concrete posits such as frictionless slopes and continuous fluids, which may play a crucial role in scientific theorizing yet to which we are not tempted to accord any ontological status. Melia, Yablo, and others have suggested parallels with metaphorical or figurative language, which may be used to convey truths while mentioning entities to which we do not wish to be ontologically committed.

The above debate came to a head in a back-and-forth series of papers by Joseph Melia ([2000], [2002]) and Mark Colyvan ([2002]), with Melia on the nominalist and Colyvan on the platonist side. Despite their opposing sympathies, both authors agreed that it is not enough—for the purposes of establishing platonism—that mathematics be indispensable for science; it has to be indispensable *in the right kind of way*. Specifically, it needs to be shown that reference to mathematical objects sometimes plays an *explanatory role* in science. Modifying the original Indispensability Argument to reflect this shift of focus yields the following 'enhanced' Indispensability Argument:

The Enhanced Indispensability Argument

(1) We ought rationally to believe in the existence of any entity that plays an indispensable explanatory role in our best scientific theories.

(2) Mathematical objects play an indispensable explanatory role in science.

(3) Hence, we ought rationally to believe in the existence of mathematical objects.

The idea, shared by both sides in the above debate, is that inserting the word 'explanatory' in premise (1) makes it more plausible because it restricts attention to cases where we can posit the existence of a given entity by inference to the best explanation.

It is worth keeping in mind that the broader strategic point of indispensability-style arguments is to avoid begging the question against the mathematical nominalist. This is the reason for the 'detour' through empirical science: more direct approaches only work if we assume the literal truth of pure mathematical claims at the outset. For example, the platonist could try establishing the existence of abstract mathematical objects by inference from claims such as 'There is a prime number between 5 and 10'. The problem is that the nominalist can escape the conclusion by accepting the inference but rejecting the premise. A similar point applies to explanation within mathematics. It may turn out that inference to the best explanation operates within mathematics, and provides a legitimate way of establishing one mathematical claim based on another. But inference to the best explanation gets no traction if the truth of the explanandum is itself in question, as it is in the nominalist/platonist debate.

The disagreement between Colyvan and Melia then comes down to whether or not premise (2) in the Enhanced Indispensability Argument is true. In other words, can convincing examples be found in which mathematical objects play an indispensable explanatory role in science? In (Baker [2005]), I have presented a putative example of mathematics playing just such an explanatory role. The example featured the life cycle of the periodical cicada, an insect whose two North American subspecies spend 13 years and 17 years, respectively, underground in larval form before emerging briefly as adults. One question raised by biologists is: why are these life cycles prime? It turns out that a couple of explanations have been given that rely on certain number theoretic results to show that prime cycles minimize overlap with other periodical organisms. Avoiding overlap is beneficial whether the other organisms are predators, or whether they are different subspecies (since mating between subspecies would produce offspring that would not be coordinated with either subspecies). The form of the explanation is sketched below:

The Cicada Example

(4) Having a life-cycle period that minimizes intersection with other (nearby/lower) periods is evolutionarily advantageous. (*biological law*)

(5) Prime periods minimize intersection (compared to non-prime periods). (*number theoretic theorem*)

(6) Hence organisms with periodic life cycles are likely to evolve periods that are prime. (‘*mixed*’ *biological/mathematical law*)

(7) Cicadas in ecosystem-type E are limited by biological constraints to periods from 14 to 18 years. (*ecological constraint*)

(8) Hence cicadas in ecosystem-type E are likely to evolve 17-year periods.

The core thesis that I defended in (Baker [2005]) is that the cicada case study is an example of an *indispensable, mathematical explanation* of a *purely physical phenomenon*. Hence, applying the Enhanced Indispensability Argument, we ought rationally to believe in the existence of abstract mathematical objects.

Since the publication of the paper presenting the cicada example, a number of responses have appeared in print objecting to various aspects of the above argument. Each of the three separate components of the core thesis mentioned above have been attacked, as well as the inference from this thesis to a broadly platonist conclusion. My principal goal in what follows is to present these objections and to discuss how the above argument might be defended against them.

3 Is the Mathematics Indispensable to the Explanation?

One way to attack the claim that the mathematics involved in the explanation of the cicada period lengths is indispensable is to show that somehow the choice of mathematical apparatus here is *arbitrary*. The thought is that if it can be shown that the choice of mathematical apparatus is just one of many equally good alternatives then the particular mathematical objects involved cannot be indispensable to the overall explanation. But what is meant by 'mathematical apparatus' here? In the mathematical portion of the cicada explanation there appear the individual numbers 13 and 17, the arithmetical property of primeness, and number theory as a whole. What is required to establish arbitrariness at these different levels varies, as do the consequences for the Enhanced Indispensability Argument of any such arbitrariness.

3.1 Object-level arbitrariness

Consider a case where an explanation of some physical phenomenon, p, cites—among other things—the fact that the two objects involved are 2 metres apart. The role of the number 2 in this explanation seems arbitrary. If asked to expand on this intuition, we might add that there seems to be nothing intrinsic to the physical nature of the situation that 'forces' the involvement of the number 2. Call this *object-level arbitrariness*.

This sort of arbitrariness is often linked in turn to the arbitrariness of the choice of units. The number 2 is involved in the explanation only because we choose to measure the distance in metres. In one sense, of course, the choice to use metres in explaining p is far from arbitrary: the metre is the standard and established unit of distance measurement in science. But had some other distance unit become established instead—for example feet, or cubits—it seems as if nothing would have been gained or lost from the explanation of p by using these units instead.

Is there object-level arbitrariness in the cicada example? Certainly the role of the numbers 13 and 17 in the explanation of cicada periods rests directly on the choice of years as the basic time units in the explanation. But in this respect the cicada example is well placed to meet any accusation of arbitrariness because the year as time-unit does seem to be bound up with genuine physical features of the situation. Years correspond to a physical regularity (one complete orbit of the Sun by the Earth); moreover, this regularity is of direct relevance to the life cycles of organisms since it affects local features such as temperature and amount of daylight.[1]

Arguably, it is not enough merely to show the physical significance and relevance of years as units in order to avoid the charge of arbitrariness. The choice of units could still be arbitrary if there are other choices of time units that are also physically significant; for example, days, lunar months, etc. Rather than pressing the point, the defender of the cicada example may do best simply to accept that the involvement of 13 and 17 in the explanation of cicada periods *is* arbitrary in some sense, but deny that object-level arbitrariness need have any adverse impact on the force of the cicada example in the context of the

[1] Even in the distance case there may be examples that avoid object-level arbitrariness. For example, if the occurrence of some physical phenomenon depends on the *ratio* of the distances between two sets of objects, then this number will be invariant under (linear) changes of distance units.

indispensability debate. For if all choices of units involve *some* mathematical objects playing an explanatory role, then one can still use inference to the best explanation to derive the conclusion that there are mathematical objects.[2]

3.2 Concept-level arbitrariness

What if we try to make the arbitrariness objection more sophisticated? Recall that the crucial notion in the proposed mathematical explanation of cicada period length is *primeness*. It is the link between primeness and minimizing intersection with other period lengths that does the explanatory work. I shall look at a couple of suggestions that have been made concerning alternative explanations of the cicada life-cycle phenomenon that do not invoke the notion of primeness. If successful, this approach lends support to the claim of *concept-level arbitrariness*.

The basic strategy is to argue that we ought to be starting with an explanandum that is more specific. Thus, Juha Saatsi writes that 'the point is that the explanandum of the biological theory is only that the periods are 13 or 17, not that the period is some *n*, where *n* is prime.'[3] If this is right, so the thinking goes, then it may open the way to alternative, non-number-theoretic explanations. One possibility is a quasi-geometrical explanation, using physical objects as 'props'. Saatsi suggests using sets of sticks of different lengths, measured out in some given unit. We could lay a series of sticks of length 13 end to end, next to another series of sticks of some other length, say 14, and see how many sticks we have to lay down before the two series are the same length. We could repeat this for other integer unit lengths close to 13, and show that 13 and 17 require the longest series of sticks compared to other nearby lengths.

A second alternative that avoids invoking primeness, suggested by Chris Daly and Simon Langford ([forthcoming]), is to seek an *intrinsic* explanation based on the precise details of each cicada subspecies' ecological past.

> 'Why . . . is their periodic life-cycle of *this* duration rather than any other?' This question focuses on the physical phenomenon of duration rather than on a mathematical theory that might be used only to index durations. The answer, supplied by evolutionary theory, will be along the following lines: given that certain relevant creatures on the cicada habitat have periodic life-cycles of some other duration, it is advantageous for the cicada life-cycle to be the particular duration it is, for this minimizes the encounters between organisms.

[2] One way to mark the difference here is in terms of order of quantifiers. We are retreating from the stronger 'EA' claim, that there are some mathematical objects that play a part in every adequate explanation of cicada periods, to the weaker 'AE' claim, that every adequate explanation involves reference to some mathematical objects.

[3] Personal communication.

There are at least two problems shared by both of these suggested lines of alternative explanation. First, they are in tension with actual scientific practice.[4] Even once biologists had good explanations for the long duration and periodicity of cicada life cycles, they remained puzzled about why these periods have the particular lengths they do. And there is good evidence, based on what they write and say, that this puzzlement only arose because of the fact that both of the known period lengths are *prime*. Second, the alternative explanations are simply not as good as the number theoretic explanation. The main reason why not is that they are not *generalizable*. Even if we can use sticks to demonstrate the optimality of 13 in one case and the optimality of 17 in the other, these separate demonstrations do not permit any predictions to be made about likely life-cycle durations in other ranges, or for other species. Similarly for explanations based on detailed ecological histories.

Useful here is David Owens' ([1992]) discussion of the notion of a *coincidence*, which he defines as a compound event whose constituent events have independent causal explanations. Consider the collision of two particles P and Q at location *l* and time *t*. There is (let us assume) a causal explanation of why particle P is at location *l* at time *t*, and a completely separate causal explanation of why particle Q is at location *l* at time *t*. But this does not explain why both particles are in the same place at the same time, thus explanation is not 'agglomerative'. Owens' focus is on causal explanation, but the notion of coincidence is applicable to explanation considered more generally. An intrinsic, historically specific explanation of why cicada subspecies A has a period of 13 years combined with a completely separate intrinsic, historically specific explanation of why cicada subspecies B has a period of 17 years does not thereby yield an explanation of why the two period lengths share the property of primeness. When biologists first posed the question, 'Why are cicada period lengths prime?', it was an open question as to what the answer would be. Indeed it was an open question whether there would turn out to be any explanation at all. Either there would be a common explanation for the primeness of the two period lengths, or there would not be. If the latter alternative transpired, the primeness of the two period lengths would have been—in Owens' terminology—a coincidence.

In addition to their shared lack of generalizability, the proposed primeness-avoiding alternative explanations also have their own separate problems. In the case of the stick explanation, the laboriousness of the method makes it difficult to be sure that we have indeed exhausted the possible arrangements of sticks, and this ought to reduce our confidence in any minimization result 'proved' by such a method. In the case of the historico-ecological approach, an explanation along these lines that is detailed enough to explain the specific

[4] Saatsi acknowledges this tension: 'It's a different question, of course, what scientists write. Perhaps the nominalist needs to point out to them that primeness *per se* is not doing any indispensable explanatory work.' (personal communication).

length of a given cicada species' period is unlikely ever to be available to us. While it might therefore be a competitor to the explanation involving primeness, it is a competitor only in theory, not in practice.[5]

3.3 Theory-level arbitrariness

If the indispensability of the notion of primeness to the (best) explanation of the cicada phenomenon is accepted, is there still any way to level a charge of arbitrariness? Perhaps, for even if it is conceded that number theoretic notions are essential to the explanation, the critic may argue that number theory does not necessarily carry commitment to numbers. By trying to break the link between number theory and numbers—construed as abstract objects—this objection amounts to a claim of *theory-level arbitrariness*. The aim is to show how nominalistic underpinnings can be provided for our number-theoretic explanations while still retaining these explanations.

It is important here to keep in mind the broader dialectical situation. Recall that both sides in this debate, the platonist (represented by Colyvan) and the nominalist (represented by Melia), accept the core premise of the original Indispensability Argument. In other words, both sides agree that mathematical objects do play an indispensable role in science. Where they disagree is on the truth of the corresponding premise in the Enhanced Indispensability Argument, namely, whether mathematical objects specifically play an indispensable *explanatory* role in science. The broader point of agreement means that both sides reject the various general strategies for nominalizing science that have appeared in the literature. Some of these strategies introduce extra operators; for example, the possibility operator of Geoffrey Hellman's ([1989]) modal structuralism. Others loosen constraints on what counts as a well-formed formula; for example, by working in a base logic that allows countably long combinations of truth-functional operators (see, for example, Melia [2001]). Others quantify over surrogates for mathematical objects; for example, the geometric strategy of Hartry Field ([1980]) based on an ontology of space-time points. In each case, a general 'recipe' is provided that—if successful—promises to eliminate mathematical objects from science. And in each case, the two crucial questions concern, first, whether the proposed framework is adequate to reproduce the functions of the platonistic mathematical theory it is replacing and, second, whether the extra apparatus invoked is nominalistically acceptable. But, for the present purposes, we do not need to resolve these issues. The Enhanced

[5] One response for the nominalist here is to argue that the theoretical availability of a historico-ecological explanation is all that matters. In other words, as long as we know that *there must be* some such explanation, that the explanation is a genuine alternative to the current mathematical explanation, and that the alternative makes no use of number theory (or other mathematics), then this is enough to undermine the indispensability of mathematics for explaining the cicada life-cycle phenomenon. For more on this line of argument, see (Melia [1995]).

Indispensability Argument is only of independent interest for those, like Colyvan and Melia, who accept the broader indispensability claim and thus deny the success of these kinds of general nominalistic reconstructions.

Therefore, any attempt to establish theory-level arbitrariness, at least in the context of the current debate, has to be more local in scope. Can it be shown that the particular parts of number theory used in the cicada example are eliminable? It is tempting to think that the number-theoretic apparatus invoked in the explanation of cicada life cycles is so basic that straightforward paraphrases will be available in the first-order logic. After all, we are familiar with cases in which apparently number-involving claims can be eliminated in this way. Thus we can avoid ontological commitment to numbers when we assert

(9) The number of F's is 2

by offering the paraphrase

(9*) $\exists x \exists y (Fx \wedge Fy \wedge x \neq y \wedge \forall z (Fz \supset (z = x \vee z = y)))$,

which avoids quantifying over numbers. In the cicada example, however, innocuous paraphrases of this sort are unlikely to be available. As has been argued above, the fact for which biologists sought an explanation involves the notion of primeness. In addition, claims involving primeness such as

(10) The number of F's is prime

cannot be paraphrased into a first-order logic so as to eliminate any mention of primeness. The reason, in a nutshell, is that there is an infinite number of ways for a number to be prime; hence, any paraphrase of (10) would have to involve an infinite disjunction of the form, 'X has life-cycle length 2 or length 3 or length 5 or length 7 or . . .'. (See Boolos [1981] for a fuller discussion.)

To summarize, I think that the cicada explanation is well placed to meet charges of arbitrariness, whether at the level of object, concept, or theory. The units involved in the explanation arise from intrinsic physical features of the situation, the number-theoretic notion of primeness plays a key role, and—despite the relative simplicity of the mathematics involved—no easy nominalistic paraphrases are available.

4 Is the Explanandum 'Purely Physical'?

Sorin Bangu attacks the cicada case study as support for platonism on the grounds that what is being explained by appeal to number theory is not a *purely physical* phenomenon:

> Baker begs the question against the nominalist. . . . If ['Cicadas have prime periods'] is taken to be true, this can't hold unless there is a mathematical

> object (specifically: a number) to which the property 'is prime' applies. Therefore, by taking the explanandum as being true (to comply with the requirements of the IBE strategy), Baker assumes realism before he argues for it. (Bangu [2008], p. 18)

Bangu's point here has some force, and before attempting a response it will be worth teasing out the links he makes to the broader nominalist/platonist debate. A key feature of the Enhanced Indispensability Argument, and one that is inherited from the original Quine-Putnam version, is that the data, for which mathematics purportedly plays a crucial explanatory role, is supposed to be the common ground for the platonist and nominalist alike. As Mary Leng has put it,

> 'Given the form of Baker's ... argument, one might wonder why it is mathematical explanations of physical phenomena that get priority. For if there are ... some genuine mathematical explanations [of mathematical facts] then these explanations must also have true explanans. The reason that this argument can't be sued is that, in the context of an argument for realism about mathematics, it is question begging. For we also assume here that genuine explanations must have a true explanandum, and when the explanandum is mathematical, its truth will also be in question.' (Leng [2005], p. 174)

Given the remarks made at the end of the previous section, my room for manoeuvre appears limited. I suggested there that the concept of primeness is unlikely to be eliminable using only non-mathematical vocabulary. Indeed this is an important part of the reason for thinking that the mathematics in the cicada explanation is indispensable. Combine this with the fact that biologists do tend to phrase the question concerning cicada period length using the concept of primeness, and it seems as if Bangu's complaint is on target. But perhaps there is a way out for the platonist. We start with two pieces of data:

(11) The length (in years) of the life cycle of cicada subspecies A is 13.

(12) The length (in years) of the life cycle of cicada subspecies B is 17.

These data are acceptable to both the platonist and the nominalist, given the possibility of paraphrasing each claim using first-order logic with identity (using the standard technique illustrated at the end of the previous section). On the basis of these data, we then advance the following theses:

(11*) The length (in years) of the life cycle of cicada subspecies A is prime.

(12*) The length (in years) of the life cycle of cicada subspecies B is prime.

The mathematical content of (11*) and (12*) cannot be paraphrased away (since '_ is prime' cannot be eliminated using standard quantifiers and identity), so they are not acceptable to the nominalist. From a philosophical perspective, therefore, we do not at this stage *endorse* (11*) or (12*) for fear of begging the question. This is the case even if biologists take them to be unproblematically true.

Next we ask whether there is an *explanation* for the tentative theses, (11*) and (12*). In discovering the number-theoretic explanation linking primeness to minimization of intersection with other period lengths, we make use of the following intermediate conclusion:

(13) The lengths (in years) of the life cycles of periodical organisms are likely to be prime.

Statement (13) yields a *common* explanation for (11*) and (12*), from which (11) and (12) follow as specific consequences once appropriate ecological constraints are introduced.

This pattern of argument combines inference to the best explanation with a form of 'bootstrapping,' but it does not involve begging the question against the nominalist. The basic data, (11) and (12), is acceptable to the nominalist. Theses (11*) and (12*) are not nominalistically acceptable, but they are initially advanced only tentatively. There are then two possible outcomes of the search for an explanation for (11*) and (12*). Either there is a common explanation of the primeness of the two period lengths or there is not.

If there is such an explanation—and the number theoretic explanation appears to be an excellent candidate—then this can *also* be turned into an explanation of (11) and (12), by adding the subspecies-specific ecological constraints. It is a good explanation because it unifies these two phenomena under a single 'argument pattern', and (relatedly) it can be generalized to other actual or hypothetical cases. For example, it predicts that other organisms with periodical life cycles are also likely to have prime periods. It is therefore better than any historico-ecological explanation that concatenates two separate and independent explanations of the two different period lengths. Hence by inference to the best explanation, we ought to believe in the entities invoked in the number theoretic explanation, which includes abstract mathematical objects such as numbers. But once numbers are included in our ontology, we need no longer be tentative about (11*) and (12*).

The pattern exemplified by the above argument is as follows:

(i) Data, D;

(ii) Tentative hypothesis, H;

(iii) Explanation, E, of H, which also can be extended to yield an explanation, E*, of D;

(iv) E* is the best explanation of D;

(v) Hence we ought to believe E*, and thereby E;

(vi) But D and E together imply H;

(vii) Hence we ought to believe H.

Alternatively, it might have turned out that there was *no* common explanation of the primeness of the two cicada periods. In this case, to use Owens' terminology from the previous section, the similarity between (11*) and (12*) would have turned out to be a *coincidence*.[6]

Viewing the overall argument in the above manner provides the platonist with an escape route from Bangu's allegations of begging the question against the nominalist. The supposition of the platonistically 'tainted' explanandum concerning primeness of cicada periods is made tentatively. And the way is left open for the explanandum to be withdrawn if no suitable explanation for it can be found.[7]

5 Is the Mathematics Explanatory in Its Own Right?

At various points in his writings, Joseph Melia has urged the view that mathematical objects are not themselves explanatory but rather that they 'index' elements of the physical situation, which themselves do the explanatory work. For example;

> '[Although we may express] the fact that a is $7/11$ metres from b by using a three place predicate relating a and b to the number $7/11$, nobody thinks that this fact holds *in virtue* of some three place relation connecting a, b and the number $7/11$. Rather, the various numbers are used merely to index different distance relations.' (Melia [2000], p. 473)

I have already argued, in the previous two sections, that the mathematical aspects of the cicada explanation are unlikely to be eliminable because of the key role played by the concept of primeness. It seems unlikely that Melia would agree that primeness really is playing a crucial role here. But even if he were to concede this point, Melia could still maintain that the mathematics is not explanatory in its own right but rather is a non-explanatory component of a larger explanation. I know of only one specific proposal for drawing a principled distinction between these two possibilities and it is to this that I now turn.

6 In fact, however, even calling this situation a 'coincidence' arguably begs the question against the nominalist here. For it suggests that there is a genuine property, having a period length (in years) that is prime, which just happens to be shared by the two cicada subspecies. The nominalist is likely to object to this, since in the absence of any common explanation she has not been given any reason to believe in the truth of (11*) or (12*).

7 Note that we could also simplify the above pattern by eliminating steps (ii) and (iii) and just move straight from data, D, to explanation E*, and thence to H as a consequence of E*. However, this fits less well with actual scientific practice (since the explanandum is typically H rather than D).

One of the earliest treatments of the topic of mathematical explanation in the philosophical literature can be found in a paper published by Mark Steiner in the late 1970s. In his article ([1978]), Steiner focuses on mathematical explanations of physical phenomena. After presenting a case study concerning the displacement of rigid bodies around fixed axes, he reaches the following conclusion:

> 'The difference between mathematical and physical explanations of *physical* phenomena is now amenable to analysis. In the former, as in the latter, physical and mathematical truths operate. But only in mathematical explanation is [the following] the case: when we remove the physics, we remain with a mathematical explanation – of a mathematical truth!' (Steiner [1978], p. 19)

If we apply the 'Steiner Test' to the periodical cicada example then what is its verdict? We immediately run up against the first problem with the Test, which is that it depends on a prior grasp of the notion of 'internal' mathematical explanation (of mathematical truths), and this is something for which there is no widely accepted philosophical account. If we rely here instead on intuitions then it would seem that the cicada explanation will probably not count as genuinely mathematical. The key mathematical results (as reproduced in Baker [2005]) are the following two lemmas:

> Lemma 1: the lowest common multiple of m and n is maximal if and only if m and n are coprime.
>
> Lemma 2: a number, m, is coprime with each number $n < 2m$, $n \neq m$ if and only if m is prime.

The proofs of these two lemmas, while relatively elementary, were not given in the paper; instead readers were referred to Edmund Landau's *Elementary Number Theory* ([1958]). My own feeling, on reviewing the proofs, is that neither is particularly explanatory. This may in part be because the results established by the lemmas in question are so basic: a few moments' reflection shows why they must be true, even without constructing a formal proof.

I conclude that the Steiner Test pronounces (albeit weakly) against the cicada explanation being a genuine mathematical explanation of a physical phenomenon. However, I don't see this as a major problem for the positive view I am defending because I think that the Steiner Test is seriously flawed. *Contra* Steiner, I would argue that the evidence from scientific practice indicates that the internal explanatory basis of a piece of mathematics is largely irrelevant to its potential explanatory role in science. For example, which shape is optimal for tiling the Euclidean plane was for a long time an open question for mathematicians. It was conjectured that the optimal shape—in the sense of minimizing total side length for arbitrarily large areas—is a regular hexagon, and this became known as the Honeycomb Conjecture. In 1999, Thomas Hales proved the Honeycomb Conjecture (Hales [2001]). The consensus among biologists seems to be that this proof *explains* why bees use hexagonal cells in their honeycombs. The form of explanation is broadly similar to that in the cicada case. A mathematical proof demonstrates the optimality of a certain solution, in this case creating the maximum number of cells with the minimum amount of material, and this is invoked as part of an evolutionary explanation for why the given behaviour occurs. However, it should be noted that the biologists do not know or care about the details—or even the general outline—of Hales' proof. It might be an explanatory proof, by mathematicians' lights, or it might not. This does not seem to affect the attribution to this mathematical result of a key explanatory role in the explanation of bees' honeycomb-building behaviour.[8] I conclude that mathematical explanations of physical phenomena do not map onto mathematical explanations of mathematical results in the neat way that Steiner claims.

In the absence of a ready-made test for the explanatoriness of a piece of mathematics in a given physical situation, we seem to have reached an impasse. A burden of proof arises here. Does the platonist need to give a positive argument for why the mathematics in the cicada case is explanatory in its own right, or does the nominalist need to give a positive argument to the contrary? Below I shall examine—though ultimately reject—an argument that the mathematics *cannot* be genuinely explanatory.

Sorin Bangu presents what he terms a 'general dilemma' for strategies that use inference to the best explanation to argue for the truth of mathematical claims based on their applicability in science. Recall that his specific objection to the cicada explanation was that the explanandum, that cicada life-cycle periods are prime, is unacceptably mathematical. In my response I argued that we can see this explanandum as initially advanced only tentatively, and that it can itself ultimately be justified as a consequence of the best explanation of facts about the specific lengths of the periods of the two cicada species' life cycles. Bangu finds this sort of move unconvincing, and complains that it ends up impaling the platonist on the second horn on Bangu's dilemma because

> 'one needs a further argument to see how the mathematical explanans can *in principle* have any explanatory relevance for an explanandum that is purely physical, free of any traces of mathematical vocabulary.' (Bangu [2008])

[8] See also (Lyon and Colyvan [2008]) who discuss the 'external' explanatory role of the mathematical result while explicitly setting aside the issue of the 'internal' explanatoriness of the proof itself.

It's unclear what sort of argument would satisfy Bangu here, but there does seem to be room for the platonist to manoeuvre. First, there are familiar cases from other contexts where facts expressed in one vocabulary are 'explanatorily relevant' to facts expressed in another, separate vocabulary. Take the observable/unobservable distinction in the philosophy of science. Facts about electrons (unobservables) can clearly help to explain facts about traces in a cloud chamber (observables), even though the latter facts can be described using claims that are 'free of any traces of electron vocabulary'.

Second, there is an equivocation in Bangu's presentation of his dilemma between two importantly distinct senses of 'mathematics-free'. Let us say that a sentence is 'mathematics-free' in a strong sense if it contains no mathematical vocabulary, and it is 'mathematics-free' in a weaker sense if it contains no *ineliminable* mathematical vocabulary. Thus,

(14) The number of cows in the field is 2

is weakly mathematics-free but not strongly mathematics-free. It contains mathematical vocabulary ('number' and '2'), but this vocabulary can be eliminated by paraphrasing (14) into first-order logic with identity. Now Bangu's earlier question-beggingness objection only gets traction if the explanandum in question contains ineliminable mathematical vocabulary. So, weakly mathematics-free statements such as (14) are permissible. And since examples of this sort do contain mathematical vocabulary (albeit eliminably), it does not seem so mysterious that mathematical theories could play a role in explanations of them.

I have now addressed two arguments against the explanatoriness of the mathematical component of the cicada explanation, one based on Steiner's 'Test' and one based on Bangu's impossibility claim. Of course, dealing with two arguments against a thesis is not in itself an argument for the thesis, and I do not know how to *demonstrate* that the mathematical component is explanatory. On the other hand, I think it is reasonable to place the burden of proof here on the nominalist. The way biologists talk and write about the cicada case suggests that they do take the mathematics to be explanatory, and this provides good grounds, at least *prima facie*, for adopting this same point of view.

6 Does Inference to the Best Explanation Apply to Mathematics?

In a paper written in 2005, Mary Leng accepts that the cicada case study provides an example of an indispensable mathematical explanation of a purely physical phenomenon. Nonetheless, she resists the move from this premise to the conclusion that we ought rationally to believe in the existence of abstract mathematical objects. Why does she do this? Leng is a scientific realist, so she accepts the legitimacy of inference to the best explanation for concrete theoretical posits. However, she rejects inference to the best explanation in the particular case of mathematics. Leng thus wishes to combine scientific realism with mathematical fictionalism.

Consider what Leng says about cases such as the cicada explanation:

> 'Couldn't a mathematical explanation get its value *as an explanation* due to the conditions it imposes on concrete, non-mathematical systems? And couldn't these conditions be imposed equally well by a fictional theory as they would be by a literally true one?' (Leng [2005], pp. 11–2)

Leng gives at least three arguments in support of taking this distinctive, fictionalist attitude towards mathematics. I shall consider them in turn.

6.1 Leng's first argument

Leng's first argument proceeds from the widely acknowledged phenomenon that fictional entities can play a useful role in scientific theories. She writes,

> 'Baker and Colyvan have (at least tentatively) accepted that our theories as a whole need not be true to be good, that they may make use of some false hypotheses in order to represent truths about physical systems...' (Leng [2005], p. 11)

Leng has in mind here familiar cases where various kinds of idealized concrete posits are introduced—frictionless slopes, continuous fluids, point masses, and so on—in the course of scientific theory construction, without scientists thereby being committed to the existence of such entities. She concludes that there is therefore room to take a similarly fictionalist line towards the mathematical apparatus of our scientific theories.

Leng is right, I think, that there is nothing incoherent *per se* about mathematical fictionalism. But her argument misses the main point of the Enhanced Indispensability Argument, which is precisely to draw a sharp line between representational and explanatory uses of mathematics. Indeed it is partly because of the sorts of cases involving idealized concrete posits that Leng mentions that the Enhanced Indispensability Argument was developed in the first place. Rejecting the existence of frictionless slopes, say, or continuous fluids, does not mean rejecting inference to the best explanation because these posits play a representational role, not an explanatory role. Leng accepts that the mathematics in the cicada case is genuinely explanatory, so her argument based on idealized concrete posits is off target.

6.2 Leng's second argument

Leng's second argument is that one can provide a model for how mathematical entities can be explanatory while nonetheless being fictional. She writes

> 'Why is it that the primeness of 13 can serve as an explanation of the cicada behaviour? Only because the succession of years is correctly modelled by the natural number system. . . . [W]e can account for the applicability of the mathematical result not as due to there actually being a number 13 or 17 with the property of primeness, but rather because it follows from the assumptions of number theory that 13 and 17 are prime.' (Leng [2005], p. 16)

Note, first, that what Leng is offering here is a kind of 'second-order' explanation. She is explaining how a mathematical explanation is possible, given fictionalism. But she is *not* using fictionalism about mathematics to explain a physical explanandum. In this situation it is crucial to distinguish between acknowledging the possible falsity of the explanans being offered and actively disbelieving in an explanans while simultaneously putting it forward as an explanation.

There are Moorean aspects here also. It does not seem odd for historians of science to explain why and how phlogiston could play an explanatory role in early modern theories of combustion, despite the fact that these historians do not themselves believe in the existence of phlogiston. Yet there would be something peculiar if someone were to explain combustion by appeal to phlogiston while simultaneously denying the existence of phlogiston. Applying this distinction back to the cicada case, an argument such as Leng's that purports to show how appeal to mathematical objects *could* be explanatory with respect to some given physical phenomena despite the mathematical objects not existing is not—or at least not obviously—an argument for suspending belief in the existence of such objects. (Just as an argument that shows how I *could* have a desk-like visual image without there being a desk in front of me is not—or at least not obviously—an argument for suspending my belief in the existence of the desk.)

6.3 Leng's third argument

Leng's third argument is that even a false mathematical theory can impose constraints on a physical system:

> [I]f true, such a theory would be true in virtue of mathematical objects being configured in a certain way and physical systems being configured in a certain way, so as to allow for the various relations posited between the mathematical and physical components to hold. The condition imposed

> on the physical world by such a theory is, as Mark Balaguer puts it, 'that the physical world holds up its end of the "empirical-science bargain"'. (Leng [2005], p. 11, footnote 4)

There are a couple of problems with this strategy. First, it invokes what is (for the fictionalist) a counterfactual, which turns on how things would be if the mathematical facts were different: 'If there were to exist mathematical objects, M, then . . .'. But just how to evaluate this sort of 'countermathematical' is far from clear.[9] Second, even if an adequate analysis of the mathematical counterfactual can be provided, this approach seems more suited to representational applications of mathematics than to explanatory applications. There may be situations in which the best description of some physical phenomenon invokes objects we believe not to exist. Consider

(15) This natural rock formation looks like a dragon's head.

If I am worried about thereby taking on a commitment to the existence of dragons, I might rephrase this descriptive claim as a subjunctive conditional:

(16) If there were dragons, then their heads would resemble this natural rock formation.

But this sort of move seems much less plausible in cases where the entities in question are introduced for *explanatory* purposes. Consider

(17) The 'missing' mass-energy in the products of the electron decay is due to a neutrino having been emitted.

Here I am explaining the missing mass-energy by positing neutrinos. The observed mass-energy does not add up *because* an (undetected) neutrino has been emitted. But I do not see how I can legitimately offer (17) as the explanation while simultaneously retreating to the weaker subjunctive conditional,

(18) If there were neutrinos then the 'missing' mass-energy in the products of the electron decay would be due to a neutrino having been emitted.

To summarize, I do not think that Leng has done enough to show that a fictionalist strategy that rejects inference to the best explanation for mathematical entities is a coherent and sensibly motivated position. I would be much more sympathetic if proponents of this style of fictionalism could come up with even one clear case where an entity (or entities) acknowledged to be fictional plays a key role in an explanation that we are inclined to accept. Unless and until this challenge is met, I think it is permissible for the platonist to disregard this line of objection.

[9] For discussion of mathematical counterfactuals in the context of a different line of argument against platonism, see (Baker [2003]). For more on the counterfactual strategy for the fictionalist, see (Dorr [2007]).

Leng is not the only philosopher who has expressed doubts concerning the applicability of inference to the best explanation to mathematical entities. In a recent paper, Juha Saatsi argues against putative parallels between indispensability arguments for mathematical realism and explanationist arguments for scientific realism. Saatsi writes:

> 'Mathematical method is deductive and cannot be assimilated with the explanation-driven inductive method that arguably rules scientific inferences. No mathematical entity has ever been *introduced* as the best explanation of some (mathematical, or physical) phenomena. No abductive inference has ever brought new mathematical facts to our attention. Mathematical knowledge . . . belongs only to the *broad* inferential background and is hence outside of the realist's justificatory gambit.' (Saatsi [2007], p. 28)

Saatsi's point seems to be that, looking at scientific and mathematical practice, it is clear that mathematical posits are not the kind of thing that are *introduced* using inference to the best explanation. Hence there are good grounds for thinking that inference to the best explanation simply does not apply to mathematical entities. How compelling is this argument?

Saatsi's first claim is that mathematical method is deductive. It is of course true that the deductive method is central to the practice of mathematics. But it is not clear that mathematical methodology is *solely* deductive. Consider the axioms of our preferred mathematical theories, for example the axioms of ZFC. However these are justified, it is not by deduction from other more basic claims. One idea is that what is going on here is abductive: the axioms are chosen that best systematize the basic set theoretical (or arithmetical, or geometrical) claims that we accept. And that doesn't look too far away from 'the inductive method'.

Saatsi goes on to argue that 'no mathematical entity has ever been introduced as the best explanation of some (mathematical or physical) phenomenon'. Let us set aside cases where the explananda are mathematical, since our concern here is with mathematics applied in science. Have mathematical entities ever been introduced to explain some physical phenomenon? Saatsi's claim seems plausible insofar as typical cases of mathematics being applied in science involve picking some mathematical theory 'off the shelf' from among those that have already been explored in purely mathematical contexts. And the cicada example is certainly a case where very familiar mathematical entities, the natural numbers, are invoked in the explanation of the physical phenomenon. Nonetheless, there are some reasons to be cautious in endorsing Saatsi's position. Firstly, it is necessary to clarify what is meant here by 'introduced'. Presumably Saatsi means 'introduced into mathematics' as opposed to 'introduced into science'. For if it is merely the latter sense that is intended then there may well be cases of familiar mathematical entities whose first appearance in a scientific context is for explanatory purposes. Even in the first sense, a detailed look at the history of science may reveal episodes where the mathematical entities in question are novel not just to science but also to mathematics. One case that comes to mind is Hamilton's introduction of quaternions in the mid-nineteenth century as an extension of complex numbers that could deal with geometrical operations in three dimensions. Quaternions were not previously known to mathematicians, hence this counts as 'introduction' in the strong sense. On the other hand, it may be the case that the role being played by quaternions in science was not primarily (or at all) explanatory. A second sort of case is where the idiosyncratic requirements of a given physical theory require a mathematical theory so specific that it has not been previously developed or investigated by mathematicians. For example, bosonic string theories require 26 spatial dimensions. It may well be that prior to its development in the context of theoretical physics, mathematicians had never specifically studied 26-dimensional geometry. Against this, a defender of Saatsi's position might respond that—even if mathematicians had never developed 26-dimensional geometry *per se*—they had developed more general geometric theories dealing with *n*-dimensional spaces, for arbitrary *n*. Hence there is nothing mathematically novel about this application to physics.

For the sake of argument, let us grant Saatsi's claim that no entities have been introduced into mathematics in order to explain some physical phenomenon. This means that mathematical entities lie (to use Saatsi's terminology) in the 'broad inferential background'. The final part of Saatsi's argument is that items in the broad inferential background cannot be justified by inference to the best explanation. But why does the prior history of an entity's role in theorizing matter to the issue of whether we ought to take on commitment to the existence of that entity in our *current* theories?

Let me illustrate my worry with a hypothetical scenario. Imagine that some maverick group of science fiction enthusiasts happened to have come up with a fairly detailed theory of black holes long before any black holes had been detected by astrophysicists or even described in principle by theoretical physicists. The science fiction enthusiasts wrote various far-fetched stories featuring black holes. And imagine that these writers got enough of the properties of black holes right for us to plausibly say that they were talking about the same concept as we now use in science. Consider the following two questions:

(i) In the above situation, would we be permitted (or even obliged) to disbelieve in the existence of black holes, since black holes were not *introduced*—in the above sense—using inference to the best explanation in science?

(ii) If so, then would there be any conceivable development of science that *would* sanction belief in the existence of black holes, given these (imagined) historical precursors?

One response to this line of objection might be to distinguish between 'existence-committing' and 'existence non-committing' prior contexts, and then argue that the prior contexts in the black-hole example were fictions, and hence no claims of their existence were literally made. Hence black holes were 'introduced' when they first appeared in science. My worry is that if this is allowed then it is going to undercut the objection against mathematical entities. We agree that the prior contexts where mathematical entities are used prior to being applied in science are nearly always in *pure mathematics*. If pure mathematics is an existence-committing context then we already seem to have a direct argument for Platonism. If pure mathematics is not an existence-committing context then, by parallel argument with the black-hole case, mathematical entities are 'introduced' when they are applied in science.

7 Conclusions

In this paper I have focus principally on the cicada case study in order to anchor the discussion and to address the specific points made by critics. However, the debate is ongoing, and there is plenty of room for further argument on both the platonist and nominalist sides.

On the platonist side, it is clearly less than ideal to rest the argument for the existence of abstract mathematical objects on a single case study from science. Thus one line of further inquiry on the platonist side is to look for more good examples of mathematical explanation in science. Not only does this have the potential to strengthen the evidence for the second premise of the Enhanced Indispensability Argument, it may help to cast light on some of the points of contention discussed above by highlighting different ways in which mathematics can play an explanatory role. For this reason it will be especially valuable to find examples involving different mathematical theories (for example, geometry) and different areas of science. It is unlikely that number theory, and biology, is the only game in town.

On the nominalist side, especially for nominalists who wish to embrace some version of fictionalism about mathematics, the most pressing area for further work is responding to the challenge I posed in the discussion of Leng's views, namely, to come up with examples of acknowledged fictions that play a genuinely explanatory role.

In their illuminating survey of nominalist strategies in the philosophy of mathematics, John Burgess and Gideon Rosen examine various ways of making the abstract/concrete distinction. One proposal they discuss links abstractness to causal inefficacy, and they raise the question of 'whether it is possible to clarify the distinctive way in which ordinary material bodies are causally active and, if so, whether it can indeed be said that . . . abstracta that are colloquially spoken of in causal terms are not causally active in that way' (Burgess and Rosen [1997], p. 23). I would argue that a parallel point can be made concerning abstracta and explanation. Taking our cue from Burgess and Rosen, we may formulate the question as follows:

> Is it possible to clarify the distinctive way in which ordinary material bodies can play an explanatory role, and if so can it indeed be said that abstracta which are mentioned in the context of scientific explanations are not explanatory in that way?

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