

**CALCULUS PLACEMENT TEST**

**Please print out, fill in your answers, and return by August 15 to:**

**Calculus Placement Test  
Department of Mathematics and Statistics  
Swarthmore College  
500 College Ave  
Swarthmore, PA 19081-1390**

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Name \_\_\_\_\_

What was your last math course and what was your grade in it? \_\_\_\_\_

What math course do you hope to take first at Swarthmore? \_\_\_\_\_

Have you taken any AP exams, or Higher Level IBs or British A-levels? If so, which exams? What was your score?

*Special Information.* What are you thinking of majoring in? (If you don't know, don't be afraid to say so.) Also, if there is anything special that you feel might affect your placement, tell us about it. For instance, tell us if you have taken one of the newer sorts of calculus courses emphasizing labs, graphing calculators, or projects; if it has been a long time since you last studied math; if you have studied mathematics beyond calculus.

**Should you study for this test?** We encourage you to review for a few hours, especially if it has been some time since you studied calculus. The Calculus Placement test covers much of the material in Swarthmore's first-year calculus sequence: Math 15 and Math 25. See the online catalog for a description of the relevant topics.

Instructions on next page

Score:            Sect    1            2            3            4

**Instructions:**

This exam is closed book but untimed. You need not take it in one sitting.

Write down in the space provided the *main steps* you use to solve each problem. If you want, first solve the problem without space restrictions on scratch paper that we will never see. For most problems, there is a line on which you should write your final answer. If you need more space for important work, you may attach extra sheets with your name on them.

**Calculators.** Except on one problem they are permitted. However, for algebraic calculations it is better on this test to show us that you can do them yourself. For instance, if you have a TI-89, it is better for you to compute derivatives than to have it do them. If you do get an answer by calculator only, say so.

The exam has 4 sections. Many of you should start with Section 1 and keep going, skipping any questions that are unfamiliar and stopping when no questions remain that you understand well enough to do. In some cases, you may start in a later section. For instance, if you have an AP score of 4 (on either the AB or BC), you may start in Section 2. For more details, see the *Math/Stat Placement* section of the *New Student Academics* web pages <http://www.swarthmore.edu/newstudentacademics.xml> prepared by the Dean's Office.

Each part of a problem is worth 5 points, unless indicated otherwise. Some partial credit will be given. A good score on Section 1 will place you out of first semester calculus (Math 15) and into Math 25. A further good score on Section 2 will place you into Math 26. Good scores on the first three sections will place you out of first-year calculus. If you get a low score on Section 1, we may ask you to take our Readiness Exam when you arrive at Swarthmore for Orientation.

Section 4 is used for placement *into* Honors second-year courses: Honors Linear Algebra (Math 28, 28S) and Honors Multivariate Calculus (Math 35). To place directly into Math 35, you also need independent placement out of some sort of linear algebra. We encourage all students with a BC 5 or equivalent to try to place into our honors courses.

To place *out of* either Math 28/28S or Math 35, you will need to take another test, downloadable from [http://math.swarthmore.edu/placement/advanced\\_work.html](http://math.swarthmore.edu/placement/advanced_work.html). Or see the bottom of the *Math/Stat Placement* section of the *New Student Academics* web pages

**SECTION 1**

1. Compute:

(a)  $\frac{d}{dx}(x \cos^2 x)$

Answer \_\_\_\_\_

(b)  $y'$  where  $y = \ln(x^2 + 2)$

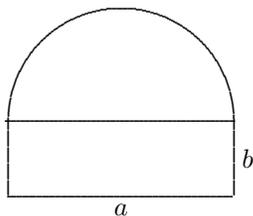
Answer \_\_\_\_\_

(c)  $\int (x^{-1/3} - \sin 3x) dx$

Answer \_\_\_\_\_

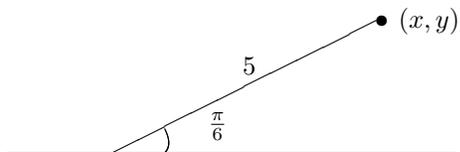
2. Find the equation of the line tangent to the curve  $y = x^5 - 2x + 1$  at the point  $(1, 0)$ . Express your answer in the form  $y = mx + b$ .

Answer \_\_\_\_\_

3. You open a savings account by making a deposit. Suppose that  $g(r)$  denotes the number of years that it takes for your account balance to grow by \$1000 if it accrues interest at an annual rate of  $r\%$  compounded continuously. What can you say about the *sign* of  $g'(r)$  — is it positive or negative? Explain briefly.4. The figure below is a rectangle topped by a semicircle. Express the total area in terms of sides  $a$  and  $b$ .

Answer \_\_\_\_\_

5. Refer to the graph below. Find the value of  $x$ . (3 points) Answer \_\_\_\_\_



6. Given  $e^{3 \ln y} = x^2 + 2$ , write  $y$  as a function of  $x$ . (3 points) Answer \_\_\_\_\_

7. Find constants  $a$  and  $b$  so that the graph of the function

$$f(x) = x^3 + ax + b$$

has a local minimum at the point  $(1/2, 1)$ . (3 points each)

$$a = \underline{\hspace{2cm}} \quad b = \underline{\hspace{2cm}}$$

8. Approximate  $\sqrt{16.3}$  with the help of the first derivative. (No calculators!)

Answer \_\_\_\_\_

9. Find the following limits. Each is worth 3 points.

(a)  $\lim_{t \rightarrow 5} \frac{9 - t^2}{5 - t}$  Answer \_\_\_\_\_

(b)  $\lim_{x \rightarrow \infty} x e^{-x} + 3 + \frac{1}{x}$  Answer \_\_\_\_\_

NOTE: Problems 10–13 refer to a certain function  $f(x)$ . The graph of its derivative  $f'(x)$  (not  $f(x)$ ) is shown in the figure at the bottom of the page.

10. For what values of  $x$  is  $f(x)$  increasing?

Answer \_\_\_\_\_

11. Where does  $f(x)$  have local extrema?

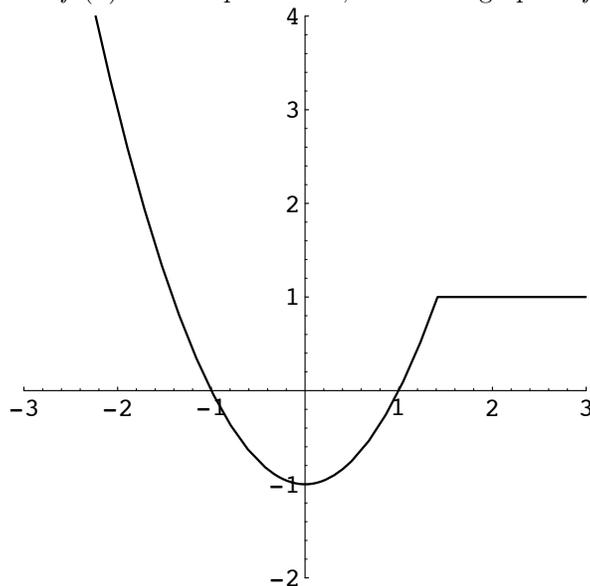
Answer: Local max at  $x =$  \_\_\_\_\_

Local min at  $x =$  \_\_\_\_\_

12. On what interval(s) is  $f(x)$  concave down? ( $\curvearrowright$ )

Answer \_\_\_\_\_

13. The graph plotted below is  $f'(x)$ . On the plot below, sketch the graph of  $f(x)$ , assuming that  $f(0) = 2$ .



14. A rectangular garden of area 75 square feet is bounded on three sides by a wall that costs \$8 per foot and on the fourth side by a fence that costs \$4 per foot. What are the most economical dimensions of the garden?

Answer \_\_\_\_\_

15. Suppose (contrary to fact) that the height  $y$  of an object thrown into the air at time 0 satisfied  $y''(t) = -12t$  m/s<sup>2</sup>. If the object was thrown upwards with a velocity of 50 m/s from a height of 5 meters, what would be its height after 2 seconds?

Answer \_\_\_\_\_

16. (a) State the limit definition of  $f'(x)$ .

(b) Use this definition to compute  $f'(x)$  when  $f(x) = x^2 + 2$ .

**SECTION 2**

1. Evaluate the following integrals:

(a)  $\int_0^1 x^2(1+2x^3)^{1/4} dx.$

Answer \_\_\_\_\_

(b)  $\int x \sin x dx$

Answer \_\_\_\_\_

(c)  $\int \frac{1}{x(x+1)} dx.$

Answer \_\_\_\_\_

2. Find the area of the region between the line  $y = \frac{1}{2}x$  and the curve  $y = \sqrt{x}$ . (First make a rough sketch.)

Answer \_\_\_\_\_

3. What is the average value of  $y = x^2$  over the interval  $0 \leq x \leq 2$ ?

Answer \_\_\_\_\_

4. If  $\int_1^4 (2f(x) + 3) dx = 7$ , what is  $\int_1^4 f(x) dx$ ?

Answer \_\_\_\_\_

5. Consider the region in the  $xy$ -plane bounded by  $y = e^x$ , the  $x$ -axis, and the lines  $x = 1$  and  $x = 3$ . This region is rotated about the  $y$ -axis, generating a solid of revolution. Set up — but do not evaluate — an integral that represents the volume of the solid.

Answer \_\_\_\_\_

6. Consider the following table of values for  $f$ :

$x$	2	3	4	5
$f(x)$	9	7	6	8

(a) Use all the data, and the trapezoid rule, to approximate  $\int_2^5 f(x) dx$ .

Answer \_\_\_\_\_

(b) Do you expect your approximation in a) to be high or to be low? Why?

7. A news broadcast in early 2005 said that the average American's annual income was changing at a rate given in dollars per month by  $r(t) = 40e^{t/500}$  where  $t$  is in months from January 1, 2005. If this trend continued, what total change in income did the average American experience during the first six months of 2005?

Answer \_\_\_\_\_

8. A circular city has radius 10 miles and population density  $1000/(1+r^2)$  people per square mile at distance  $r$  miles from the center. What is the population of the city?

Answer \_\_\_\_\_

9. Evaluate  $\int_{-1}^1 x^{-2} dx$ , or show that it diverges.

Answer \_\_\_\_\_

**SECTION 3**

1. Do the following sequences and series converge or diverge? Give reasons for your conclusions.

(a)  $\sum_{k=0}^{\infty} \left(\frac{4}{5}\right)^k$

Answer \_\_\_\_\_

Reasons:

(b) the sequence  $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$

Answer \_\_\_\_\_

Reasons:

(c)  $\sum_{n=1}^{\infty} \frac{n^2}{n!}$

Answer \_\_\_\_\_

Reasons:

(d)  $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

Answer \_\_\_\_\_

Reasons:

2. Starting from the Taylor series around  $x = 0$  for  $e^x$  and  $\cos x$ , compute the 2nd-degree Taylor polynomial around  $t = 0$  for  $e^{2t} \cos t$ .

Answer \_\_\_\_\_

3. (a) Determine the 2nd-degree Taylor polynomial around  $x = 1$  for  $\sqrt{x}$ .

Answer \_\_\_\_\_

- (b) Give a bound on the error, valid in the range  $\frac{1}{2} \leq x \leq \frac{3}{2}$ , when you approximate  $\sqrt{x}$  using your 2nd degree polynomial.

Answer \_\_\_\_\_

4. The mystery function  $f(x)$  has the Maclaurin series

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k+1}.$$

- (a) What is the radius of convergence?

Answer \_\_\_\_\_

- (b) What is  $f''(0)$ ?

Answer \_\_\_\_\_

- (c) What is the Maclaurin series for  $f'(x)$ ?

Answer \_\_\_\_\_

- (d) Express  $\int_0^1 f(x) dx$  as an infinite series of numbers.

Answer \_\_\_\_\_

#### SECTION 4

INSTRUCTIONS: This section is for students who place out of calculus and wish to take *Honors* second-year courses: Math 28 or 28S, Honors Linear Algebra, or Math 35, Honors Multivariate Calculus. Math 35 requires linear algebra as well as this section of this test. To place *out of* either 28/28S or 35 you need to take another test, as explained at the end of the instructions on page 2.

Attach your work on additional sheets and put your name on each one.

These questions are fairly close in style to some of those in 28 or 35, but they are not on material covered in those courses. The point is: you should find questions like these interesting and be able to get a good start on them, even if you have never seen anything like them. The value of this section for *you* is to help you decide if you like the flavor of our Honors courses. We will also make a recommendation based on how well you do. Each problem is worth 15 points.

1. Suppose that a function  $f$ , defined for all real numbers, satisfies the property that, for all  $x$  and  $y$ ,

$$f(x + y) = f(x) + f(y). \quad (*)$$

- (a) (2 points) Name a function that satisfies property (\*). Name another that doesn't. Justify your answers.
- (b) (3 points) Prove that any function that satisfies property (\*) also satisfies  $f(3x) = 3f(x)$ .
- (c) (5 points) Prove that any function that satisfies property (\*) also satisfies  $f(x - y) = f(x) - f(y)$ .
- (d) (5 points) Suppose in addition to (\*) that  $f$  is differentiable at  $x = 0$ . Prove that  $f$  is differentiable everywhere.
2. A **field** is any set of number-like things which obey all the usual laws of high school algebra (commutativity, distributivity, etc.), and such that when you add, subtract, multiply or divide any pair of them (except you can't divide by 0), you get *another thing in the same set*. In other words, a field has to be *closed* under these 4 operations. For instance, the positive integers are not a field; they are closed under addition and multiplication, but not subtraction and division. Sure,  $4 - 7$  is a number, but it is not in the set of positive integers. On the other hand, the set of all real numbers is a field.

Prove or disprove that the following are fields, by proving or disproving that they are closed under the four operations (except for division by 0). (The other requirements like the commutativity law, are satisfied automatically because each of these sets is a subset of the real numbers.)

- (a) (4 points) The set of integers
- (b) (5 points) The set of rational numbers
- (c) (6 points) The set of numbers  $r + q\sqrt{2}$ , where  $r$  and  $q$  are any rationals.

3. For  $n = 1, 2, \dots$  define the function  $f_n$  with domain  $[0, 1]$  as follows:

$$f_n(x) = \begin{cases} 4n^2x & \text{if } 0 \leq x \leq \frac{1}{2n}, \\ 4n - 4n^2x & \text{if } \frac{1}{2n} \leq x \leq \frac{1}{n}, \\ 0 & \text{if } x \geq \frac{1}{n}. \end{cases}$$

Finally, define  $f$  by

$$f(x) = \lim_{n \rightarrow \infty} f_n(x).$$

- (a) (3 points) Graph  $f_1$  and  $f_3$ .
- (b) (4 points) For each  $x$  in  $[0, 1]$ , what is the value of  $f(x)$ ? Why?
- (c) (4 points) Compute  $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx$ .
- (d) (2 points) Compute  $\int_0^1 (\lim_{n \rightarrow \infty} f_n(x)) dx$ .
- (e) (2 points) Do you find anything interesting about your answers to the previous parts?
4. For any positive integer  $n$ , let  $S_n$  be the collection of all subsets of  $\{1, 2, \dots, n\}$ , including the empty set  $\emptyset$ . We define an operation  $+$  on sets as follows:

$$A + B = \{\text{all elements in exactly one of } A \text{ and } B\}.$$

For instance,

$$\{1, 3\} + \{2, 3, 7\} = \{1, 2, 7\}.$$

Furthermore,  $A + B + C$  is defined as  $(A + B) + C$ . It is true (though you need not show) that  $(A + B) + C = A + (B + C)$ .

Explain why  $S_n$  under  $+$  satisfies the following properties:

- (a) (3 points) There is a “zero”. That is, there is a set  $Z$  in  $S_n$  (you have to identify it), so that for all  $A$  in  $S_n$ ,  $A + Z = A$ .
- (b) (3 points) There are inverses. That is, for every set  $A$  in  $S_n$ , there is a set  $A'$  in  $S_n$  (you have to identify it), so that  $A + A' = Z$ .
- (c) (5 points) Every set in  $S_n$  is the sum of zero or more of the sets  $\{1\}$ ,  $\{1, 2\}$ ,  $\{1, 2, 3\}$ ,  $\dots$ ,  $\{1, 2, 3, \dots, n\}$ .
- (d) (4 points) Finally, determine if every set in  $S_5$  can be written as the sum of zero or more of the sets  $\{1, 2\}$ ,  $\{2, 3\}$ ,  $\{3, 4\}$ ,  $\{4, 5\}$  and  $\{5, 1\}$ .