Real Analysis 1

(1) Show that if $f : [0,1] \to [0,1]$ is continuous, then there is a point $x_0 \in [0,1]$ such that $f(x_0) = x_0$.

(2) Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\delta > 0$. Show that $F(x) = \dfrac{1}{2\delta} \displaystyle\int_{x-\delta}^{x+\delta} f(t)\,dt$ is differentiable and find its derivative.

(3) Show that the set $\displaystyle\bigcup_{n\in\mathbb{N}} \left[n + \frac{1}{n}, n + 1 - \frac{1}{n}\right]$ is closed. Is it compact?

(4) Let V_0 be the space of continuous functions on $[0,1]$ that are differentiable on $(0,1)$ with continuous and bounded derivative and that satisfy $f(0) = 0$. Show that

$$d(f,g) = \int_0^1 |f'(x) - g'(x)|\,dx$$

defines a metric on V_0.

(5) Show that if $f, g : [a,b] \to \mathbb{R}$ are Riemann integrable so is the product fg.

Real Analysis 2

(1) Assume that f is Lebesgue measurable on $\mathbb{R}$. Show that if $\int_E f \geq 0$ for all measurable E, then $f(x) \geq 0$ for almost all x.

(2) Show that if f is measurable then $|f|$ is measurable. Does the reverse also hold? Explain.

(3) Assume that A is a bounded measurable subset of $\mathbb{R}$ of measure 0. Show that $A^2 = \{x^2 : x \in A\}$ is also of measure 0. Is it necessary to have A bounded?

(4) Let H be the parallelogram in $\mathbb{R}^2$ whose vertices are $(1,1)$, $(3,2)$, $(4,6)$, $(2,5)$. Find the affine map T which sends $(0,0)$ to $(1,1)$, $(1,0)$ to $(3,2)$, $(1,1)$ to $(4,6)$, $(0,1)$ to $(2,5)$. Show that $J_T = 7$. Use T to convert the integral

$$\beta = \int_H e^{x-y}$$

to an integral over $[0,1] \times [0,1]$ and thus compute β.

(5) Let $k > 1$. Let M be a compact, oriented k-manifold in $\mathbb{R}^n$. State conditions under which the formula

$$\int_M f\, d\omega = \int_{\partial M} f\omega - \int_M (df) \wedge \omega$$

is valid.