## Real Analysis 1

(1) Show that if $f : [0,1] \to [0,1]$ is continuous, then there is a point $x_0 \in [0,1]$ such that $f(x_0) = x_0$.

(2) Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\delta > 0$. Show that $F(x) = \dfrac{1}{2\delta} \displaystyle\int_{x-\delta}^{x+\delta} f(t)\, dt$ is differentiable and find its derivative.

(3) Show that the set $\displaystyle\bigcup_{n \in \mathbb{N}} \left[ n + \frac{1}{n}, n + 1 - \frac{1}{n} \right]$ is closed. Is it compact?

(4) Let $V_0$ be the space of continuous functions on $[0,1]$ that are differentiable on $(0,1)$ with continuous and bounded derivative and that satisfy $f(0) = 0$. Show that

$$d(f,g) = \int_0^1 |f'(x) - g'(x)|\, dx$$

defines a metric on $V_0$.

(5) Show that if $f, g : [a,b] \to \mathbb{R}$ are Riemann integrable so is the product $fg$.

## Complex Analysis

*In this part of the exam you may not use Picard's Theorem. Any other result from your class can be used as long as you name the result and verify all needed conditions.*

(1) How many points are in the solution set of the equation $e^{-z} = Re(z)$?

(2) Show that if the sequence of complex numbers $\{a_n\}$ satisfies $\displaystyle\sum_{n=1}^{\infty} |a_n| < \infty$, then $f_N(z) = \displaystyle\sum_{n=1}^{N} a_n z^n$ converges uniformly to a holomorphic function on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$.

(3) Show that if $f$ is analytic in a domain $D$ and takes only real values, then $f$ must be constant.

(4) Find the value of the indefinite integral $\displaystyle\int_{-\infty}^{\infty} \frac{\cos^2(t)}{1+t^2}\, dt$.

(5) Assume that $f$ is entire and that $\alpha$ is not in the image of $f$. Show that either $\alpha$ is a limit point of the image of $f$ or $f$ is constant.