Swarthmore College Mathematics
Honors Examination -Topology 2005

Divide your efforts approximately equally among parts I, II and III.

Unless otherwise specified, if using a known theorem, be sure to state it carefully before using it.

I. Point-set topology

1.a. If $f : [0,1] \to R$, the real numbers, what can you say about f that depends on f being continuous? No proofs required.

1.b. If $g : R \to [0,1]$ is continuous, which of the properties of f apply also to g? For those that don't, provide a counter-example.

2.a. Prove or disprove: An arbitrary product ΠX_α of spaces X_α is connected if and only if the spaces X_α are. If one of the implications is valid but not the other, give respectively a proof and a counter example.

2.b. Prove or disprove: An arbitrary product ΠX_α of spaces X_α is path connected if and only if the spaces X_α are. If one of the implications is valid but not the other, give respectively a proof and a counter example.

3. Name conditions on a space X such that if A and B are disjoint closed subspaces of X, then any continuous function from the disjoint union $A \amalg B$ to $[0,1]$ can be extended to all of X.

II. Algebraic topology of surfaces

4.a. Define the Euler characteristic and the genus of a compact surface without boundary.

4.b. The connected sum $X\#Y$ of two distinct compact surfaces without boundary X and Y is defined by removing an open disk from each of X and Y and then identifying the newly created boundaries. Relate the Euler characteristic and the genus of $X\#Y$ to those of X and Y. What happens if the two distinct disks are removed from a single compact surface X?

5.a. Describe and explain the fundamental group of an arbitrary compact surface without boundary.

5.b. Describe and explain the fundamental group of an arbitrary compact surface after a point is removed.

5.c. Explain the relation between 5.a and 5.b in terms of the van Kampen theorem.

6. Describe all possible covering spaces $X \to Y$ where $X = P^2$ is the projective plane and $Y = T^2$ is the torus and, vice versa, when $Y = P^2$ is the projective plane and $X = T^2$ is the torus. Explain, quoting any theorems you need.

III. General algebraic topology

7. Compute the homology of a pretzel (Philadelphia style = solid 2-hole torus) using the basic properties of homology and your knowledge (without having to prove) of the homology of a circle and of a point.

8. Let $p : X \to Y$ be a map and F a discrete space with more than one point such that there is a collection of open sets U_α covering Y with homeomorphisms $h_\alpha : U_\alpha \times F \to p^{-1}U_\alpha$ such that $p \circ h_\alpha = \pi_\alpha$, the projection of $U_\alpha \times F$ onto U_α.

8.a. Prove or disprove: $p : X \to Y$ is a covering space of Y.

8.b. Prove or disprove: $p : X \to Y$ has the unique path lifting property.

8.c. Describe a (generally non-trivial) homomorphism from the fundamental group of $\pi_1(Y)$ (with any base point) to the group of homeomorphisms $F \to F$.

9. Determine the fundamental group of the following spaces V, W and Z and explain your reasoning.

V is the result of taking a solid torus (i.e. the product of a circle and a disk) and removing a 'core' (i.e. the result is the product of a circle and an annulus).

W is the result of taking a solid torus and removing a cylinder (the product of an open disc and an interval) transverse to the circle direction and thin enough so as to not disconnect the space. See the picture below.

Z is the result of taking V and removing a cylinder transverse to the circle direction as for W and positioned so as to provide two tunnels to the core, one on each side thereof.

Here is a sketch of W.