SWATHMORE COLLEGE
DEPARTMENT OR MATHEMATICS AND STATISTICS
TOPOLOGY HONORS EXAMINATION 2004

You may skip problems 3 and 6.

Use without proof the basic theorems that you learned such as van Kampen's theorem, the Lefschetz fixed point theorem and the classification of surfaces. Also use what you know about the fundamental group and homology groups of surfaces, including the Klein bottle and the torus.

Each problem has explanations and hints [in brackets].

1) Suppose that $f : X \to Y$ is a continuous map of topological spaces where X is compact but not necessarily Hausdorff and Y is Hausdorff. Then show that the graph of f is a compact and closed subset of $X \times Y$. [The *graph* of f is the set of all $(x, y) \in X \times Y$ so that $y = f(x)$. Compact does not imply closed since $X \times Y$ might not be Hausdorff.]

2) Let $A \subseteq \mathbb{R}^2$ be the union of the x-axis, the y-axis and the graph of the function $y = 1/x$:

$$A = \{(x, y) \in \mathbb{R}^2 \mid xy(xy - 1) = 0\}$$

Show that A is not connected. [Hint: Find a continuous function $f : A \to \mathbb{R}$ whose image is not connected.]

3) [Alternate problem] Suppose that B is a compact Hausdorff space which is uncountable (i.e., has more than a countable number of elements). Let U be the set of all $x \in B$ having a countable neighborhood. Show that the complement of U in B is uncountable. [You may use the fact that any finite or countably infinite union of countable sets is countable.]

4) This problem concerns connected two-fold coverings $\widetilde{S} \to S$ of unoriented surfaces S.

(a) What is the relationship between the Euler characteristics of S and $\widetilde{S}$?

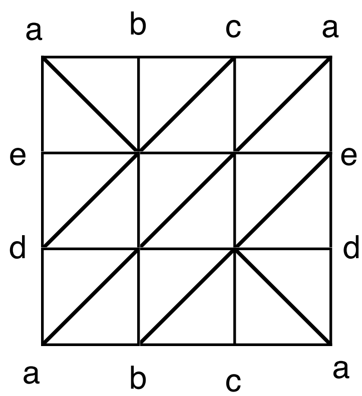
(b) Show that, up to homeomorphism, there are exactly two connected surfaces which are two fold coverings of the Klein bottle. [You may use the well-known fact that any connected nonorientable surface has a unique connected oriented two-fold covering.]

(c) Extend this theorem to any connected unoriented surface S which is not projective space. [Hint: Express S as a connected sum of two unoriented surfaces.]

5) Suppose that K is a finite simplicial complex and f is a simplicial involution of K, i.e., a simplicial bijection $f : K \to K$ so that $f \circ f = id_K$. Let $|f| : |K| \to |K|$ be the induced map on geometric realizations. [I am using the formal definition that K is a finite set whose elements are simplicies and $|K|$ is the union of the geometric simplices.]

(a) Show that the fixed point set of $|f| : |K| \to |K|$ is $|L|$ where L is a subcomplex of the first barycentric subdivision sdK of K. [Hint: The vertices of L are the fixed points of f.]

(b) Show that the Lefschetz number of $|f|$ is equal to the Euler characteristic of L.



6) [Alternate problem. This problem asks you to compute π_1, H_1 and H_2 of a torus with one additional 2-cell attached.] Let T be the torus with the above triangulation with vertices and edges identified as indicated. [In any simplicial complex the 1-simplices (edges) are determined by their endpoints. So, the top edges ab, bc, ca are identified with the bottom edges.] Let X be the simplicial complex obtained from T by adding one extra 2-simplex with vertices a, b, c (and edges ab, bc, ac). [I.e., X is T with the "hole" filled in with a triangular sheet of paper.]

(a) Compute $\pi_1 X$.

(b) Compute $H_1 X$ and $H_2 X$.

You may use the standard facts about $\pi_1 T$ and $H_*(T)$.

7) Suppose that the following diagram commutes and has exact rows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

Then show that the following induced sequence of kernels is exact.

$$0 \to \ker f \to \ker g \to \ker h$$

8) Compute the homology of the chain complex:

$$0 \to \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^4 \xrightarrow{\partial_1} \mathbb{Z}^2 \to 0$$

where ∂_2 sends the generator of $\mathbb{Z}$ to $(1,1,1,1) \in \mathbb{Z}^4$ and $\partial_1(a,b,c,d) = (a+b-2c, a-b)$. [Hint: $a+b-2c$ and $a-b$ have the same parity.]