Honors Exam in Topology
Swarthmore College
Department of Mathematics and Statistics, 2003

The exams contains 12 problems. Do at least 2 problems from each section. In each section, the harder problems are first, and will be marked appropriately.

GENERAL TOPOLOGY:

I.1 Let K be a compact subset of a topological space X. Is $\overline{K}$ (the closure of K) also compact? [Prove or provide a counterexample.]
Can you give a condition on X such that if $K \subseteq X$ is compact, it is automatically also closed?

I.2 Let the taxi cab metric
$$d_t : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$$
be defined by $d_t((a,b),(c,d)) = |a-c| + |b-d|$.
(a) Prove that this is indeed a metric.
(b) Prove that it induces the same topology on $\mathbf{R}^2$ as that induced by the usual metric.

I.3 Take the space $X = [0,1]/\sim$ where $0 \sim 1$ and all other points are only equivalent to themselves. Prove that this is homeomorphic to
$$S^1 = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}.$$

SURFACES:

II.1 (a) Describe pictorially the process of adding a handle to a surface.
(b) Explain why adding a handle reduces the euler characteristic of the surface.

II.2 Define the euler characteristic of a combinatorial surface.
Let K be a compact combinatorial surface. Let K^1 be the first barycentric subdivision of K. Show that K and K^1 have the same euler characteristic.

II.3 Let K be a combinatorial surface and T a maximal tree in K. Prove that the euler characteristic of K is 2 if and only if the dual graph to T is also a tree.

HOMOTOPY and FUNDAMENTAL GROUP:

III.1 Prove that the fundamenal group of the torus,
$$T = [0,1] \times [0,1]/\sim$$
(where $(0,y) \sim (1,y)$ and $(x,0) \sim (x,1)$ and there are no other relationsh) is isomorphic to $\mathbf{Z} \times \mathbf{Z}$.

III.2 State, and prove, the Brouwer Fixed Point theorem.

III.3 (a) Define path composition $\alpha \cdot \beta$ (assuming $\alpha(1) = \beta(0)$).
(b) Define what it means for two paths to be homotopic relative to $\{0,1\}$.

(c) Show that

$$(\alpha \cdot \beta) \cdot \gamma \simeq_{\{0,1\}} \alpha \cdot (\beta \cdot \gamma)$$

where $\simeq_{\{0,1\}}$ means homotopic relative to $\{0,1\}$.

ALGEBRAIC TOPOLOGY:

IV.1 Denote real projective m-space by P^m.
- (a) Define P^m.
- (b) Calculate the fundamental group of P^m.

IV.2 (a) Give a triangulation for the Möbius band.
- (b) Prove that the Möbius band is homotopy equivalent to the circle, S^1.

IV.3 Calculate the homology groups of P^2.