1998 Honors Examination in Statistics
(for students having taken Stat 53 and Stat 111)

Department of Mathematics and Statistics
Swarthmore College

Instructions: This exam consists of a total of four questions. Number the questions clearly in your work and start each question on a new page. You must show your work to make it clear how you obtained your answers. Answers without any work may lose credit even if they are correct, and will receive no credit if incorrect.

This is a closed-book three-hour exam. You may not refer to your notes or textbooks.

1. You plan to purchase 100 radios from a manufacturer, intending to sell them at your department store. If 10 or more of the radios are defective, you will be unable to make a profit. You decide to inspect a sample of 8 radios for defects before purchasing the entire lot of 100 radios. Let X be the number of defective radios you find in the sample. Also, let p be the proportion of the original 100 radios that are defective.

   (a) What is the exact distribution of X?

   (b) State a null and alternative hypothesis in terms of p for testing whether there are 10 or more defective radios in the 100 you plan to purchase.

   (c) Suppose you decide to reject the null hypothesis if two or more radios in the sample of eight are defective. Find the largest probability of a type I error for this procedure. You may leave your answer in the form of an unsimplified expression.

   (d) Again, suppose you decide to reject the null hypothesis if two or more radios in the sample of eight are defective. If exactly 10 of the 100 radios are defective, what is the probability of a type II error? Again, you may leave your answer in the form of an unsimplified expression.

2. In this day and age, it's difficult enough for a man to find the woman of his dreams, but it's that much more difficult if he is overweight. At the "National Heffer Dating Service," potential dates are chosen in an unusual manner. Two fair coins labeled A and B are flipped. Let X denote the total number of heads appearing, and let $Y = 1$ if heads appears on coin A, and $Y = 0$ otherwise. The choice of a date will depend on the coin flips, as described below.

   (a) Display the joint probability mass function of X and Y in a suitable table, and determine the marginal distributions of X and Y.

   (b) Determine $\mathrm{Cov}(X, Y)$.

   (c) Let W be the weight (in pounds) of the potential date. If both $X = 1$ and $Y = 1$, then a date is chosen such that $W \sim \mathrm{N}(175, 25^2)$. Otherwise, a date is chosen such that $W \sim \mathrm{N}(230, 15^2)$. Write down an expression for $\mathrm{P}(W > 200)$, the (unconditional) probability that the man is set up with a woman who weighs over 200 pounds, in terms of unconditional probabilities involving a standard normal random variable, $Z \sim \mathrm{N}(0, 1)$.

   (d) Let p denote the probability of being set up with a woman weighing over 200 pounds. In terms of p, what is the probability that, on the 8th visit to the dating service, a man is set up with a fifth woman weighing over 200 pounds?

3. A random variable X has a probability mass function

$$f(x;\theta) = \begin{cases} \frac{1}{4}(1-\theta) & \text{if } x = 0,1,2,3 \\ \theta & \text{if } x = 4 \\ 0 & \text{otherwise,} \end{cases}$$

   where θ, $0 \le \theta \le 1$, is an unknown parameter. We wish to estimate θ based on a sample of $n = 50$ independent observations $X_1, X_2, \ldots, X_{50}$.

   (a) Let

$$Y = \text{number of } X_i \text{ out of 50 that are equal to 4.}$$

   What is the exact distribution of Y, given θ?

(b) Show that $\hat{\theta}_1 = Y/50$ is an unbiased estimator of θ.

(c) Let

$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_{50}}{50}$$

be the sample mean of the 50 observations. Find numbers a and b so that the estimator $\hat{\theta}_2 = a\bar{X} + b$ is an unbiased estimator of θ.

(d) Find the variances of $\hat{\theta}_1$ and $\hat{\theta}_2$. Based on the variances of these unbiased estimators of θ, which estimator would you recommend using? Briefly justify.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with probability density

$$f(x;\theta) = \begin{cases} (\theta+1)x^{\theta} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

for $\theta > -1$.

(a) Derive the method-of-moments estimator of θ.

(b) Sometimes method-of-moments estimators can produce nonsense estimates, depending on the observed data (for example, in normal models, method-of-moment estimates of variances are sometimes negative). Does the estimator you derived in part (a) ever produce nonsense estimates of θ? If so, give an example of data coming from the above distribution that would produce a nonsense estimate. If not, show why not.

(c) Derive the maximum likelihood estimator of θ.

(d) Determine the minimum variance of an unbiased estimator of θ given by the Cramér-Rao lower bound.