Honors Examination in Real Analysis and Differential Geometry
May 2003

Answer as many questions as you can. State clearly any results you rely upon. Make your responses brief but complete; explain your reasoning, and write clearly.

1. According to the Heine–Borel theorem, every closed, bounded set X of real numbers $\mathbb{R}$ is compact, in the sense that a cover of X by open sets always has a finite subcover.

   Show that any open interval $I = (a, b)$ in $\mathbb{R}$ fails to be compact by constructing a cover $\mathcal{U}$ for I that has no finite subcover. Demonstrate why $\mathcal{U}$ has no finite subcover.

2. (a) Prove that the sequence of real numbers

   $$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$$

   has a limit L and $0 \le L < 1$.

   (b) Prove the more subtle result that $\frac{1}{2} \le L$.

3. Consider the following two arguments, each of which begins with a valid statement. In the first case, the conclusion is correct, and the method is valid. The second case uses the same method, but reaches the absurd conclusion $\ln 2\sqrt{2} = \ln 2$.

   What justifies the first argument—that is, why does it work—and why does the apparently similar second argument fail?

$$\begin{aligned}
\frac{\pi^2}{6} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \cdots \\
\frac{\pi^2}{24} = \frac{1}{2^2}\frac{\pi^2}{6} &= \quad + \frac{1}{2^2} \qquad + \frac{1}{4^2} \qquad + \frac{1}{6^2} \qquad + \cdots \\
\frac{\pi^2}{8} = \frac{\pi^2}{6} - \frac{\pi^2}{24} &= 1 \qquad + \frac{1}{3^2} \qquad + \frac{1}{5^2} \qquad + \frac{1}{7^2} + \cdots
\end{aligned}$$

$$\begin{aligned}
\ln 2 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \cdots \\
\ln\sqrt{2} = \frac{1}{2}\ln 2 &= \quad + \frac{1}{2} \qquad - \frac{1}{4} \qquad + \frac{1}{6} \qquad - \frac{1}{8} \qquad + \frac{1}{10} \qquad - \frac{1}{12} + \cdots \\
\ln 2\sqrt{2} &= 1 \qquad + \frac{1}{3} - \frac{2}{4} + \frac{1}{5} \qquad + \frac{1}{7} - \frac{2}{8} + \frac{1}{9} \qquad + \frac{1}{11} - \frac{2}{12} + \cdots \\
&= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots \\
&= \ln 2
\end{aligned}$$

4. (a) What condition must a metric space satisfy to be *complete*?

   (b) Consider the space $C[0,1]$ of continuous functions $f(x)$ with $0 \le x \le 1$, first with the max norm, and then with the L_1 norm:

   $$\|f\|_{\max} = \max_{0 \le x \le 1} |f(x)| \qquad\qquad \|f\|_{L_1} = \int_0^1 |f(x)|\,dx.$$

   In each case, the norm induces a metric $d(f,g) = \|f - g\|$ on $C[0,1]$. In each case, determine whether the metric is *complete*. If it is, explain why, using standard facts of real analysis. If it is not, give a counterexample.

5. Suppose $d(x,y)$ is a metric on the set X. Let
$$\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}.$$
Show that $\rho(x,y)$ is also a metric on X, and show that the metric spaces (X,d) and (X,ρ) have the same open sets.

6. Let (Y,d) be a compact metric space and $\mathcal{C}$ a collection of closed subsets of Y. Suppose every finite subcollection of closed sets from $\mathcal{C}$ has a non-empty intersection; prove that the entire collection has a non-empty intersection.

7. Suppose $\varphi(t,y)$ is a continuously differentiable function defined on an infinite strip $a \le t \le b$, $-\infty < y < \infty$ containing the point $(t,y) = (p,q)$ in its interior.

   Let $C[a,b]$ be the space of continuous functions $f(t)$ on $[a,b]$ with the maximum norm, and let $M : C[a,b] \to C[a,b]$ be the map defined by
$$Mf(t) = q + \int_p^t \varphi(s, f(s))\, ds.$$

   (a) Show that there are numbers a^* and b^*, with $a \le a^* < p < b^* \le b$, such that the map M, when restricted to the space $C[a^*,b^*]$ with the max norm, is a contraction mapping:
$$\|Mf\|_{\max} \le \mu \|f\|_{\max}, \qquad 0 \le \mu < 1.$$

   (b) Explain why your result in part (a) implies the initial value problem $y' = \varphi(t,y)$, $y = q$ when $t = p$ has a unique solution $y = g(t)$ on the interval $a^* < t < b^*$. That is, show
$$g'(t) \equiv \varphi(t, g(t)), \quad a^* < t < b^*$$
$$g(p) = q.$$

8. Determine the value of the surface integral
$$\iint_S (\mathbb{V}\cdot \mathbf{n})\, dA,$$
where S is the unit sphere, $\mathbf{n}$ is the outward unit normal on S, and $\mathbb{V} = \left(x - \frac{1}{2}, y, \frac{1}{3}z\right)$.

9. Describe, in words, the image of the Gauss map of an ordinary torus (with circular core and circular cross-section). What aspect of the Gauss map indicates that the total curvature of the torus is zero.

10. (a) What are the *elliptic*, *parabolic*, and *hyperbolic* points on a smooth surface embedded in $\mathbb{R}^3$.

    (b) Consider a surface of revolution S in $\mathbb{R}^3$ parametrized in the form
$$x = f(s)\cos\theta, \qquad y = f(s)\sin\theta, \qquad z = g(s),$$
where $f > 0$ and g are smooth functions and s is the arc-length parameter on the generating curve $C : (x,z) = (f(s), g(s))$. Identify two different types of points on C that generate circles of parabolic points on the surface S.

    (c) Prove your assertion in part (b).