Swarthmore College
Honors Exam in Real Analysis

MAY 2006

Instructions: Do as many of the following problems as you can. Justify all answers. You may quote any standard result as long as that result is not essentially what you are being asked to prove. If you do quote a standard result, make sure your clearly identify the result and verify that the hypotheses are satisfied.

In these problems, a function f is said to be of class C^k if f is continuous and all of its (ordinary or partial) derivatives up through order k exist and are continuous. It is said to be of class C^∞ if it is of class C^k for every $k \geq 0$.

1. For what real values of p is $d_p(x, y) = |x - y|^p$ a metric on $\mathbb{R}$?

2. Suppose $\{f_n\}$ is a sequence of continuous functions on $[0, 1]$, and let $F_n(x) = \int_0^x f_n(u)\, du$ for $x \in [0, 1]$. If the functions f_n are uniformly bounded, show that some subsequence of $\{F_n\}$ converges uniformly on $[0, 1]$.

3. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function that is periodic of period 2π, meaning that $f(t+2\pi) = f(t)$ for all $t \in \mathbb{R}$. Let a_n and b_n be the Fourier cosine and sine coefficients of f, defined by

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\, dt, \qquad n \geq 0,$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\, dt, \qquad n > 0.$$

   (a) If f is of class C^1 on $\mathbb{R}$, prove that there is a constant $C > 0$ such that

$$|a_n| \leq \frac{C}{n}, \quad |b_n| \leq \frac{C}{n}, \qquad \text{for all } n > 0.$$

   (b) Now suppose that there are constants $C > 0$ and $\alpha > 2$ such that

$$|a_n| \leq \frac{C}{n^\alpha}, \quad |b_n| \leq \frac{C}{n^\alpha}, \qquad \text{for all } n > 0.$$

   Prove that f is of class C^1.

4. Consider the following series for $x > 0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{x+n}.$$

   Show that the series converges to a differentiable function $f\colon (0, \infty) \to \mathbb{R}$ with

$$f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(x+n)^2}.$$

5. Suppose f is a continuous real-valued function on $[0, \infty)$. Define a sequence of functions $f_1, f_2, \ldots : [0, \infty) \to \mathbb{R}$ by

$$f_1(x) = \int_0^x f(t)\, dt,$$
$$f_2(x) = \int_0^x f_1(t)\, dt,$$
$$\vdots$$

and in general,

$$f_{n+1}(x) = \int_0^x f_n(t)\, dt.$$

(a) Prove that for each $n \geq 1$,

$$f_n(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t)\, dt.$$

(b) Prove that the series $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely on $[0, \infty)$ and express it in terms of an integral with no summation.

6. Define a subset $S \subset \mathbb{R}^4$ by

$$S = \{(x, y, u, v) : x^2 + y^2 - 2uv = x^2 - y^2 + u^2 - v^2 = 0\}.$$

(a) Prove that there exist open sets $U, V \subset \mathbb{R}^2$ such that $(-1, 1, 1, 1) \in U \times V$, and differentiable functions $\alpha, \beta \colon U \to \mathbb{R}$ with the following property: $(x, y, u, v) \in S \cap (U \times V)$ if and only if $(x, y) \in U$, $u = \alpha(x, y)$, and $v = \beta(x, y)$.

(b) Compute the following partial derivatives at $(x, y) = (-1, 1)$:

$$\frac{\partial \alpha}{\partial x}, \quad \frac{\partial \alpha}{\partial y}, \quad \frac{\partial \beta}{\partial x}, \quad \frac{\partial \beta}{\partial y}.$$

7. Let $U \subset \mathbb{R}^3$ be the open set

$$U = \{(x, y, z) : x^2 + y^2 + z^2 > \tfrac{1}{2}\},$$

and let η be the 2-form on U defined by

$$\eta = \frac{x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

(a) Show that η is closed.

(b) Prove that there is no C^∞ closed 2-form on all of $\mathbb{R}^3$ that is equal to η on U.

8. Let M denote the metric space whose elements are the C^∞ functions $f \colon [0, 1] \to \mathbb{R}$, with metric

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [0, 1]\}.$$

Define maps $D \colon M \to M$ and $I \colon M \to M$ by $D(f) = f'$ and $I(f) = F$, where

$$F(x) = \int_0^x f(t)\, dt.$$

Show that I is continuous but D is not.