Swarthmore College
Honors Exam

Real Analysis and Partial Differential Equations Examination
Spring 1999

Please do as many problems as you can. Give concise proofs, showing all of your work. You may quote and use standard results, but inadequately supported answers will receive little or no credit. Good luck!

1. Let (M, d) be a metric space, and suppose that $f : M \to \mathbf{R}$ is a continuous function. Prove that $\{x \in M \mid f(x) = 0\}$ is a closed subset of M.

2. Prove, or give a counterexample: if $F_1 \supseteq F_2 \supseteq \cdots$ is a decreasing sequence of compact subsets of a metric space, then $\bigcap_{n=1}^{\infty} F_n$ is nonempty, and consists of a single point.

3. Consider the metric space $(\mathbf{Z}, |\cdot|)$, the integers with the usual absolute value.

(i) Describe the open sets in this metric space.

(ii) Show that there are no infinite compact subsets in this metric space.

4. Let (M, d) be a metric space, and $x_o \in M$. Define $f : M \to \mathbf{R}$ by $f(x) = d(x, x_o)$.

(i) Prove that f is uniformly continuous on M.

(ii) Let A be a compact subset of M. Using part (i), or otherwise, show that there exists a point $y_0 \in A$ such that

$$d(y_o, x_o) = \min_{y \in A} d(y, x_o)$$

5. Consider the sequence of functions $f_n(x) = x^n$, defined on $[0, 1]$. Determine the pointwise limit $f(x)$, and determine whether $f_n \to f$ uniformly on $[0, 1]$. Prove that $f_n \to f$ uniformly on $[0, \delta]$, where $0 < \delta < 1$.

6. (i) Let $\{f_n\}$ be a sequence of real-valued continuous functions on the interval $[a, b]$. Prove that if $f_n \to f$ uniformly on $[a, b]$, then

$$\lim_{n \to \infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$

(ii) Show by example that the above conclusion does not hold if $\{f_n\}$ is merely assumed to converge pointwise to f.

7. (i) Let $f : [a, b] \to [0, \infty)$ be continuous. Prove that if $\int_a^b f(x)\, dx = 0$, then $f(x) = 0$ on $[a, b]$.

(ii) Is the result in (i) true if f is merely assumed to be Riemann integrable?

8. (i) State a form of the Fundamental Theorem of Calculus on the interval $[0,1]$.

(ii) Let $\{f_n\}$ be a sequence of real-valued, continuously differentiable functions defined on $[0,1]$; that is, for each n, f_n is continuous, with continuous derivative f_n' on $[0,1]$. Assume that $f_n \to f$ uniformly on $[0,1]$, and that there exists a function g on $[0,1]$ such that $f_n' \to g$ uniformly on $[0,1]$. Prove that f is then continuously differentiable, and that $f' = g$ on $[0,1]$.

9. Prove that the following series defines a continuous function on $\mathbf{R}$:

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1} \sin nx.$$

10. Suppose that the Fourier transform of the function $f(x)$ is given by $\hat{f}(\lambda) = \frac{1}{1+\lambda^2}$. Find the Fourier transform of $f'(x)$, and of $f(x+2)$.

11. (i) For the following partial differential equation, identify those regions in the xt-plane where the solution $u(x,t) = 0$; you need not explicitly solve the PDE:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad x, t \in \mathbf{R} \\ u_t(x,0) &= 0 \\ u(x,0) &= \chi_{[1,2]}(x), \end{aligned}$$

where $\chi_{[1,2]}(x)$ is the function that is equal to one on the interval $[1,2]$, and zero elsewhere.

(ii) Discuss what you think is meant by the statement "for the wave equation, discontinuities in the initial data are propagated along the characteristic lines." Support your discussion with at least one example.

12. A rod of length π is initially at temperature zero throughout. The end at $x = 0$ is kept at α degrees, while the end at $x = \pi$ is kept at β degrees.

(i) Write a partial differential equation with initial and boundary values to model this situation.

(ii) What behavior do your predict for the solution as $t \to \infty$?

(iii) Solve the PDE in (i), and show that your solution demonstrates the behavior predicted in (ii).

13. Assume that $u(x,t)$ is a solution of the diffusion equation

$$\begin{aligned} u_t &= u_{xx}, \quad x \in \mathbf{R}, t > 0 \\ u(x,0) &= \phi(x), \end{aligned}$$

and that there exist constants m, M such that $m \leq \phi(x) \leq M$ for all $x \in \mathbf{R}$. Prove that $m \leq u(x,t) \leq M$ for all $x \in \mathbf{R}, t > 0$.