Swarthmore College Honors Exam
Real Analysis and Partial Differential Equations
May 4, 1995

This examination is three pages long and has 3 parts. Please complete 3 of the 4 questions in each part.

Part 1.

1. Suppose that $f(x)$ is a continuous function on the real line and that for all real numbers a and b,

$$\int_a^b f(x)\,dx = 0$$

Prove that $f(x) = 0$ for all x.

2. Let $f(x, y)$ be continuously differentiable on the square $0 \leq x \leq 1, 0 \leq y \leq 1$. Prove that

$$\frac{\partial}{\partial x} \int_0^1 f(x,y)\,dy = \int_0^1 \frac{\partial f(x,y)}{\partial x}\,dy.$$

3. Let $f(x)$ be a continuous function on the interval $[0, 1]$ satisfying $|\,f(x)\,| \leq 1$. Define $F(x)$ on $[0, 1)$ by:

$$F(x) = \sum_{n=0}^{\infty} \int_0^x f(y)y^n\,dy.$$

   a. Show that the series converges for all $x \in [0, 1)$, so $F(x)$ is well-defined.

   b. Prove that $F(x)$ is continuous on $[0, 1)$.

   c. Prove that $\int_0^1 |\,F(x)\,|\,dx$ is well-defined and finite.

4. Let $\{f_n(x)\}$ be a sequence of continuous functions on $(-\infty, \infty)$ which c formly to zero. Suppose, in addition, that $|\,f_n(x)\,| \leq x^{-2}$ for all x.

   a. Prove that

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(x)\,dx = 0.$$

   b. Give a counterexample to show that the conclusion of a. may not "uniformly to zero" is replaced by "pointwise to zero."

5. Let $f : \mathbf{R}^2 \to \mathbf{R}^3$ and $g : \mathbf{R}^3 \to \mathbf{R}^2$ be given by:

$$f(x) = (e^{2x_1+x_2}\,,\ 3x_2 - \cos x_1\,,\ x_1^2 + x_2 + 2)$$
$$g(y) = (3y_1 + 2y_2 + y_3^2\,,\ y_1^2 - y_3 + 1)$$

a. Let $F(x) = g(f(x))$. Compute $DF(0)$.

b. If p is near $(5, 0)$, is there a unique q near $(0, 0)$ so that $F(q) = p$? Why?

6. a. What is the formula for the volume, V, of a k-parallelopiped defined by vectors $x_1, x_2, ..., x_k$ in $\mathbf{R}^n$?

b. Show that the formula gives the usual definition of volume of a parallelogram in the plane.

c. Suppose $k \leq n$ and let A be an open set in $\mathbf{R}^k$. Suppose that $\alpha : A \to \mathbf{R}^n$ is of class C^r, $r \geq 1$, and let $Y = \alpha(A)$. Explain why

$$Vol\,(Y) = \int_A V(D\alpha)$$

is a reasonable formula for the volume of Y.

7. Denote the k-tensors and alternating k-tensors on a vector space V by $L^k(V)$ and $A^k(V)$, respectively.

a. If V and W are vector spaces and $T : V \to W$ is a linear transformation, define the dual transformation $T^* : L^k(W) \to L^k(V)$ and show that it maps $A^k(W)$ into $A^k(V)$.

b. Describe the spaces $A^1(\mathbf{R}^3), A^2(\mathbf{R}^3), A^3(\mathbf{R}^3), A^4(\mathbf{R}^3)$.

c. Let $f \in A^2(\mathbf{R}^5)$ and $g \in A^1(\mathbf{R}^5)$ be given by the formulas $f(\mathbf{x}, \mathbf{y}) = x_2 y_5 - y_2 x_5$ and $g(\mathbf{x}) = x_4$ where the subscripts denote components in the standard basis. Compute $f \wedge g$.

8. a. Carefully state the generalized Stokes' theorem for compact oriented k-manifolds in $\mathbf{R}^n$.

b. Show how the classical Green's theorem in the plane can be derived from Stokes' theorem (explaining clearly how the abstract objects in Stokes' theorem are related to vector fields and classical integrals).

Part 3.

9. Write a short essay discussing the basic facts about Fourier series. What is a Fourier series? For what kinds of functions does the Fourier series converge? Not converge? What is the Parseval relation?

10. A thin metal bar with insulated sides is lying on the x axis from $x = 0$ to $x = L$. The temperature, $u(x,t)$, at each x at each time t satisfies:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \kappa \frac{\partial^2 u}{\partial x^2} \\ u(x,0) &= f(x) \\ \frac{\partial u(0,t)}{\partial x} &= 0 \\ \frac{\partial u(L,t)}{\partial x} &= 0. \end{aligned}$$

a. What is the physical meaning of the boundary conditions at $x = 0$ and $x = L$?

b. Derive a series representation for the solution, $u(x,t)$.

c. What happens as $t \to \infty$? Why is that reasonable?

d. Explain why the solution is an infinitely differentiable function of x and t for $t > 0$ even if f has jump discontinuities.

11. a. Write down the wave equation for the displacement of an infinitely long elastic string lying along the x axis.

b. Write down the d'Alembert solution in terms of the initial displacement and velocity of the string.

c. Suppose the initial velocity is zero everywhere and the the initial diplacement is zero everywhere except the intervals $[-2,-1]$ and $[0,1]$. Where can you guarantee that the solution is zero?

12. Using the Fourier transform, solve the following initial value problem on the line. (β and κ are positive constants.)

$$\begin{aligned} \frac{\partial u}{\partial t} + \beta u &= \kappa \frac{\partial^2 u}{\partial x^2} \\ u(x,0) &= f(x) \end{aligned}$$