Swarthmore College
Department of Mathematics and Statistics
Honors Examination: Algebra
Spring 2023

Instructions: This exam contains nine problems. Try to solve *six* problems as completely as possible. Do not be concerned if some portions of the exam are unfamiliar; you have a number of choices so that you do not need to answer every question. Once you are satisfied with your responses to six problems, make a second pass through the exam and complete as many parts of the remaining problems as possible. I am interested in your thoughts on a problem even if you do not completely solve it. In particular, submit your solution even if you cannot do all the parts of a problem. When there are multiple parts, you are permitted to address a later part without solving all the earlier ones.

1. Let G be a finite group and let p be a prime dividing the order of G.

(a) Prove that G contains an element of order p.
(b) Show that the number of elements in G of order p is not congruent to 1 modulo p. Is there an easy way to fix this statement?

2. Consider $\alpha = \sqrt{2} + \sqrt{3}$.

(a) Determine the minimal polynomial of α over $\mathbb{Q}$.
(b) Is the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ Galois?
(c) Determine the Galois group $\mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$.

3. For each of the following rings, determine whether the ring is a UFD, PID, Euclidean domain, or a field. That is, for each part below, you are making four determinations.

(a) $\mathbb{Z}[\sqrt{-5}]$
(b) $\mathbb{Z}[i]$
(c) $\mathbb{Z}[\sqrt{2}]$
(d) $\mathbb{Q}[\sqrt{2}]$

4. In this exercise, we will consider groups of order 24.

(a) Classify all abelian groups of order 24.
(b) Compute the center of the dihedral group

$$D_{24} = \langle x, y \mid x^{12} = y^2 = xyxy = 1 \rangle.$$

(c) Show that there are no simple groups of order 24.

5. Let F be a finite field of order p^n for some prime p and positive integer n.

(a) We say that a field F is *algebraically closed* if every non-constant polynomial over F has a root in F. Prove that F cannot be algebraically closed.

(b) Construct the smallest possible field extension of GF(2) that contains both GF(2^4) and GF(2^6) as subfields.

6. Let R, S be commutative rings with identity, and let $f : R \to S$ be a ring homomorphism.

(a) Prove that if S is a domain and f is injective, then $\ker(f)$ is a prime ideal of R.

(b) Find an example of rings R and S, a ring homomorphism $f : R \to S$, and a prime ideal P of R such that $f(P)$ is a nonzero ideal of S that is not prime. (Hint: Try $R = \mathbb{Z}$.)

(c) Find an example of a ring R, a field S, and a ring homomorphism $f : R \to S$ such that $\ker(f)$ is not a maximal ideal of f. (Hint: Try $R = \mathbb{Z}[x]$.)

(d) Find an example of rings R and S, a ring homomorphism $f : R \to S$, and a prime ideal P of R such that P is a prime ideal of R and $f(P)$ is a maximal ideal of S, but P is not a maximal ideal of R. (Hint: Try $R = \mathbb{Z}[x]$.)

7. Consider the dihedral group of order 8 given by

$$D_8 = \langle x, y \mid x^4 = y^2 = xyxy = 1 \rangle.$$

(a) What are the conjugacy classes of D_8?

(b) Find a two-dimensional irreducible real representation of D_8. Prove that your representation is irreducible.

(c) Prove that all the other irreducible real representations of D_8 are one-dimensional.

(d) Considering complex representations, construct the character table for D_8.

8. Let R be a commutative ring with identity, and let $I_1, I_2, \ldots, I_n$ be ideals of R.

(a) Prove that the product of these ideals is contained in their intersection; that is,

$$I_1 I_2 \cdots I_n \subseteq I_1 \cap I_2 \cap \cdots \cap I_n.$$

(b) Is it true that the quotient ring $R/(I_1 \cap I_2 \cap ... \cap I_n)$ is an integral domain if and only if each of the quotient rings R/I_i is an integral domain?

9. Determine whether each of the groups below is finite or infinite. In case it is finite, produce the order itself and a description of the group (e.g., it is cyclic of order 10, dihedral of order 8, etc.).

(a) $G = \langle x, y \mid x^2 = y^2, xy = yx \rangle$

(b) $G = \langle x, y \mid x^2 = y^2 = [x, y]^2 = 1 \rangle$, where $[x, y] = xyx^{-1}y^{-1}$

(c) $G = \langle x, y \mid x^4 = y^3 = 1, xy = y^2 x \rangle$.

(d) $G = \langle x, y \mid x^2 = y^2, x^3 = y^3 \rangle$