Swarthmore College
Department of Mathematics and Statistics
Honors Exam in Geometry 2023

Complete as many of the questions as you can; there is no expectation that you complete all problems. Feel free to concentrate your efforts on 3 of the 4 main topics.

1 The Sphere

1. Consider the round 2-sphere embedded in $\mathbb{R}^3$. A chart (or coordinate system) on the sphere is given by

$$(\theta, \phi) \colon (0, \pi) \times (0, 2\pi) \to S^2$$

where θ measures the angle down from the z-axis and ϕ is the polar angle in the xy-plane.

   (a) Which points on the sphere are not in the image of this chart?

   (b) Why can't this chart be onto the sphere? (hint: consider the topology (or homeomorphism class) of the domain and range)

2. The standard (round) Riemannian metric with respect to these coordinates is

$$g = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}$$

which can also be written in the form $ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$.

Consider the following calculation of the volume of the sphere (when we say volume of a 2-manifold we mean surface area)

$$V = \int \sqrt{|det(g)|} \, d\theta \wedge d\phi = \int_0^{2\pi} \int_0^{\pi} \sin\theta \, d\theta \, d\phi = 4\pi.$$

Consider the following rectangles which have equal volume in parameter space

$$(0, \pi/4) \times (a, b) \qquad \text{and} \qquad (\pi/4, \pi/2) \times (a, b)$$

Do these rectangles have equal volume on the sphere? Explain why or why not.

3. Now you will use the limit comparison definition of Gaussian curvature

$$k = \lim_{r \to 0^+} 12 \frac{\pi r^2 - A(r)}{\pi r^4}$$

to calculate the curvature of the round unit 2-sphere at the north pole.

   (a) Let $A(r)$ denote the intrinsic area of a geodesic ball on the manifold (in this case the round unit 2-sphere with the coordinate chart and metric as above). Calculate the intrinsic area $A(r)$ of a geodesic ball centered at the point $(0, 0)$, i.e. the intrinsic area of the parameter domain $(0, r) \times (0, 2\pi)$.

   (b) Evaluate the limit expression for Gaussian curvature using your value of $A(r)$. (hint: you may find L'Hopital's rule or a Taylor expansion of $A(r)$ helpful)

2 The Sphere Theorem

Consider a compact simply connected Riemannian manifold M^n whose sectional curvature K satisfies

$$0 < hK_{max} < K \leq K_{max}.$$

The Sphere Theorem says: If $h = 1/4$ then M^n is homeomorphic to a sphere.

4. Prove the Sphere Theorem for dimension $n = 2$.

5. We define complex projective space $\mathbb{C}P^n$ to be the space consisting of all complex lines in $\mathbb{C}^{n+1}$, i.e. the quotient of $\mathbb{C}^{n+1}\backslash\{0\}$ by the equivalence relation relating all complex multiples of each point together. The standard metric on $\mathbb{C}P^n$ is the Fubini-Study metric where sectional curvature is given by

$$K(\sigma) = 1 + 3\langle JX, Y\rangle^2$$

   where $\{X, Y\} \in T_p\mathbb{C}P^n$ is an orthonormal basis of the 2-plane σ and $J\colon T_p\mathbb{C}P^n \to T_p\mathbb{C}P^n$ is the complex structure on $\mathbb{C}P^n$ (similar to multiplication by i).

   (a) What is the range of sectional curvatures for $\mathbb{C}P^n$ with this Fubini-Study metric?

   (b) Which 2-planes σ achieve the max sectional curvature and which achive the min sectional curvature?

   (c) What does this example say about the Sphere Theorem?

   (d) $\mathbb{C}P^1$ is a closed 2-dimensional real manifold, i.e. a closed surface. What can you say about the topology (or homeomorphism class) of $\mathbb{C}P^1$.

3 Surface Curvature

Given a surface M^2 we can find a unit normal vector $N(p)$ that this perpendicular to M^2 at p. The **Gauss map** $N\colon M^2 \to S^2$ is defined locally by recording the direction of the normal. One can define curvature of surfaces by measuring the rate at which $N(p)$ changes.

The derivative of a map between manifolds is a linear map between their tangent spaces. The derivative

$$dN_p : T_pM^2 \to T_{N(p)}S^2$$

admits representation as a symmetric 2×2 matrix, called the **shape operator**.

6. Preliminaries

   (a) What is an example of a surface for which the Gauss map can NOT be defined globally?

   (b) Why does dN_p admit a representation as a 2×2 matrix?

   (c) Why does the matrix dN_p admit a diagonalization?

   (d) What is the interpretation of the eigenvalues and eigenvectors of dN_p?

7. The **Gaussian curvature** of M^2 at p is given by $k(p) = \det(dN_p)$ and the **mean curvature** is given by $H(p) = \frac{1}{2}\text{tr}(dN_p)$. The **total Gauss curvature** is given by $\int_M k\, dA$ and the **total mean curvature** by $\int_M H\, dA$. The **Wilmore energy** W of a surface M^2 is defined as $W(M) = \int_M H^2\, dA$. Recall for a 2-sphere of radius R that the shape operator can be expressed (in an eigenbasis) as

$$dN_p = \begin{bmatrix} 1/R & 0 \\ 0 & 1/R \end{bmatrix}$$

   Compute the total Gauss curvature, the total mean curvature, and the Wilmore energy for a round 2-sphere of radius R.

8. The Wilmore Conjecture (which has been proved) states: If M^2 is a closed surface in $\mathbb{R}^3$ with genus greater than zero, then $W(M) \geq 2\pi^2$. Prove or provide a counterexample to the following statement: If S is a closed surface in $\mathbb{R}^3$ with $W(S) \geq 2\pi^2$ then the genus of S is positive.

4 Hyperbolic Space

9. Consider the Poincaré half-plane model of hyperbolic geometry, i.e. the manifold

$$\mathbb{H} = \{(x, y) : y > 0;\ x, y \in \mathbb{R}\}$$

together with the metric

$$g = \begin{bmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{bmatrix}$$

(a) Find an atlas for this manifold. (hint: this should be trivial)

(b) What does this atlas imply about the topology (or homeomporphism class) of the manifold?

Recall that the length of a curve is given by the following formula:

$$L(c) = \int_a^b |\dot{c}(t)|\, dt = \int_a^b \sqrt{g_{c(t)}(\dot{c}(t), \dot{c}(t))}\, dt.$$

(c) Find the length of the curve $c(t) = \langle t, 0.1 \rangle$ for $1 \leq t \leq 3$

(d) Find the length of the curve

$$s(t) = \begin{cases} \langle 1, t \rangle & 0.1 \leq t \leq 1 \\ \langle t, 1 \rangle & 1 \leq t \leq 3 \\ \langle 3, 4 - t \rangle & 3 \leq t \leq 3.9 \end{cases}$$

(e) Sketch both these curves. Which has shorter intrinsic length? Explain.

(f) Does there exist a shorter curve between this pair of endpoints?

(g) (bonus) What is the shortest curve (or geodesic) between the endpoints?