Swarthmore College Honors Exam - Combinatorics - Spring 2006

Instructions:
- This is closed book, notes etc.
- Pick two problems to work on from each of the following 'groups' of problems 1-4, 5-8, 9-12.
- Provide concise and well thought out answers. If there is a part of answer where you are not sure of what to do, it is better to admit the gap then to make a mistake or fudge your answer.
- Many of the questions have more straightforward or computational parts followed by more theoretical parts. Be sure to answer some of the theoretical parts.
- If you spend more than 30 minutes on a problem you may be spending too much time on it. Make sure that you move on if you are stuck on a part of a problem.
- The problems are not of equal difficulty. You will not be evaluated based on some number of correct answers but rather on demonstrating the ability to write clear, concise and correct answers to a reasonable variety of problems of different types and levels of difficulty.

1: The Ramsey number $R(p,q)$ is the smallest number r such that every two coloring of the edges of K_r, the complete graph on r vertices contains either a red clique of size p or a blue clique of size q.

(a) Show $R(3,3) = 6$.
(b) Prove that $R(p,q)$ exists and $R(p,q) \leq \binom{p+q-2}{p-1}$.
(c) The Ramsey number $R_k(3)$ is the smallest number r_k such that every k-coloring of the edges of K_r contains a triangle (a clique of size 3) in one of the colors. Prove that $R_k(3) \leq k(R_{k-1}(3) - 1) + 2$.

2: Recall that the chromatic number $\chi(G)$ of a graph is the minimum number of colors needed to color the vertices of G so that adjacent vertices get different colors. Recall also that a planar graph is a graph that can be drawn in the plane without crossing edges.

(a) Prove that $\chi(G) \leq \Delta(G) + 1$, $\chi(G) \leq n(G) - \alpha(G) + 1$ and $\chi(G) \geq n(G)/\alpha(G)$ where $\Delta(G)$ is the maximum degree of a vertex in G, $n(G)$ is the number of vertices and $\alpha(G)$ is the maximum size of an independent set in G.
(b) Prove Euler's formula $n-e+f = 2$ for connected planar graphs where n, e, f are, respectively, the numbers of vertices, edges and faces of G.
(c) Use Euler's formula to prove that a simple planar graph has a vertex with degree at most 5.
(d) Outline (you do not need to provide all of the details) a proof that every planar graph has chromatic number at most 5. You may use part (c).

3: Recall that a tournament is an orientation of the edges of a complete graph.

(a) Prove Redei's Theorem: Every tournament of order n contains a path of length $n-1$. (This is called a Hamiltonian path.)
(b) A transitive tournament is a tournament whose vertices can be labelled $1, 2, \ldots, n$ such that the arcs are (i,j) for $i < j$. Prove that every tournament that is not transitive contains a directed cycle on 3 vertices.
(c) Prove that every tournament contains a king, a vertex z such that for every other vertex x either (z,x) is an arc or there exists a y such that (z,y) and (y,x) are arcs.

4: Recall the Konig-Egervary Theorem: For a bipartite graph G, the maximum size of a matching in G is equal to the minimum size of a vertex cover of the edges of G. Equivalently, if A is a $0,1$ matrix, the maximum number of 1's, no two on a line is equal to the minimum number of lines needed to cover the 1's.
Recall also Hall Theorem: A bipartite graph G with bipartition X, Y has a matching covering X if and only if for each $S \subseteq X$ we have $|N(S)| \geq |S|$ where $N(S)$ is the neighborhood of S, the set of all vertices adjacent to a vertex of S. Equivalently, sets $X_1, X_2, \ldots, X_m$ have a system of distinct representatives if and only if for all $k = 1, 2, \ldots, m$, the union of any k of the sets has size at least k.

(a) Explain the 'equivalently' in the descriptions above.
(b) Prove the Marriage Theorem: A regular bipartite graph has a matching saturating all the vertices. Equivalently, a $0,1$ matrix for which all row and column sums equal the same constant contains a permutation matrix.
(c) Do *one* of the following: Prove the Konig-Egervary Theorem, Prove Hall's Theorem, derive Hall's Theorem from the Konig-Egervary Theorem, derive the Konig-Egervary Theorem from Hall's Theorem.
(d) Prove the Birkhoff-Von Neuman Theorem: If A is an $n \times n$ doubly stochastic matrix then A is a convex combination of permutation matrices: $A = \sum_{k=1}^{m} \lambda_i P_i$ where the P_i are permutation matrices and the λ_i are non-negative numbers summing to 1.

5: Recall that the binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ count the number of size k subsets of $\{1, 2, \ldots, n\}$.

(a) Prove the statement above about what the binomial coefficients count.
(b) Prove Pascal's identity: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
(c) Prove the Binomial Theorem for positive integers n: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
(d) Use the Binomial Theorem to prove that the number of odd size subsets of $\{1, 2, \ldots, n\}$ equals the number of even size subsets.
(e) Give a combinatorial proof (*not* using the Binomial Theorem) of part (d).

6: Recall that the Fibonacci numbers are given by the recursion $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = F_1 = 1$.

(a) Use induction to prove that $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$.
(b) Use the recursion to derive the (ordinary) generating function for Fibonacci numbers $F(x) = \frac{1}{1-x-x^2}$
(c) Use the generating function from part (b) to give another proof of the formula in part (a).
(d) Explain why the Fibonacci numbers F_n count the number of sequences of 1's and 2's with sum n.
(e) Prove that $\sum_{k=0}^{n} \binom{n-k}{k} = F_n$ (where $\binom{a}{b} = 0$ if $b > a$).

7: Recall the inclusion-exclusion formula in the following form: If $A_1, A_2, \ldots, A_n$ are subsets of a universe such that the size of the intersection of any k of the sets is independent of the choice of the sets and has size g_k (with g_0 interpreted to count the size of the universe) then the number of elements in none of the sets is given by $\sum_{k=0}^{n}(-1)^k\binom{n}{k}g_k$.

(a) At a small school with 100 students four courses are offered. Each class has 40 students, for every pair of classes there are 20 students attending both classes, for every group of 3 classes there are 10 students attending all 3 and there are 5 students taking all 4 classes. How many students are not taking any classes?
(b) Use inclusion-exclusion to solve the drunk professor problem: A drunk professor returns papers at random to a class of n students. In how many ways can this be done so that no student gets their own paper back?
(c) How many ways can the drunk professor return a test and a quiz to each student so that no student gets both of their own papers back (they might get one of the two).
(d) Explain why the inclusion-exclusion formula holds. You may use the result of 5(d).

8: A weak composition of a positive integer k with n parts is a non-negative integral solution to $\sum_{i=1}^{n} x_i = k$. A partition of a positive integer k is a non-negative integral solution to $\sum_{j=1}^{\infty} j\alpha_j = k$.

(a) Show that the number of weak compositions of k with n parts, that is, the number of non-negative integral solutions to $\sum_{i=1}^{n} x_i = k$, is equal to the number of k element multisets from $\{1, 2, \ldots, n\}$ and that this number is $\binom{n+k-1}{k}$.
(b) Show that the number of partitions of k, that is, the number of non-negative integral solutions to $\sum_{j=1}^{\infty} j\alpha_j = k$, is the number of multisets of positive numbers with sum k and give the ordinary generating function $P(x) = \sum_{k=0}^{\infty} P_k x^k$ for partitions (where P_k is the number of partitions of k).
(c) Determine the number of weak compositions of k with n parts such that each part has size at least 1 (which is the number of compositions of k with n parts). Determine the number of weak compositions of k with n parts such that each part has size at least t for some given t.
(d) Prove that the number of partitions of k into parts of size at most t is equal to the number of partitions of k into at most t parts.

9: Recall that in Polya's enumeration theory that the pattern inventory is the generating function with the coefficient of $c_1^{\alpha_1} c_2^{\alpha_2} \cdots c_t^{\alpha_t}$ equal to the number of different colorings with color i used α_i times. Here different means in different orbits under the group action under consideration.

(a) Explain how the pattern inventory can be obtained from the cycle index polynomial.
(b) Give the pattern inventory for the number of necklaces with 4 beads (free to move in space) using two possible colors of beads. (This is the same as coloring either the edges or vertices of a square free to move in space).
(c) If instead there are three possible colors for the beads in part (b) how many different necklaces are there in total?

10: Recall that an $n \times n$ matrix A is irreducible if the rows and columns can be permuted so that A has block form $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ where A_{11} is $k \times k$ for some $0 < k < n$. Equivalently, a matrix is irreducible if and only if D is strongly connected where D is the digraph with adjacency matrix $A = A(D)$. Recall also that for a digraph D, its adjacency matrix $A(D)$ has entry $A(i, j)$ equal to the number of arcs directed from i to j. A strongly connected digraph is primitive if the greatest common divisor of the lengths of closed directed walks is 1. Equivalently, a digraph is primitive if some power of the adjacency matrix $A(D)$ has all entries positive.

(a) Explain the equivalence between the digraph and matrix definitions of irreducible given above.
(b) Explain the equivalence between the digraph and the matrix definitions of primitive given above.
(c) Prove that A is irreducible if and only if every entry of $(I + A)^{n-1}$ is positive.
(d) If A is primitive its spectral radius ρ is positive and A has a positive eigenvector. Explain, in terms of theorems that you know, why this is true.
(e) If A is doubly stochastic and primitive show that its spectral radius is 1.

11: Recall that a Hadamard matrix of order n is an $n \times n$ matrix with entries from $\{-1, +1\}$ such that $H^T H = I_n$ where I_n is the identity matrix.

(a) Prove that the order of a Hadamard matrix is either 2 or a multiple of 4.
(b) Describe how to construct infinitely many Hadamard matrices.
(c) Show that there is a Hadamard matrix of order $n = 4t \geq 8$ if and only if there is a symmetric $2 - (4t - 1, 2t - 1, t - 1)$ design. (See problem 12 for notation for designs.)

12: Recall that a $2-(v, k, \lambda)$ design is a collection of b size k subsets of a v element set such that every element appears in r of the subsets and every pair of elements appears together in λ subsets. The elements are also sometimes called varieties or points and the sets are often called blocks. If $k < v - 1$ these are called balanced incomplete block designs (BIBD's). It is symmetric if $b = v$. For a symmetric BIBD with blocks $B_1, B_2, \ldots, B_v$ the derived design with respect to B_1 has elements (varieties, points) B_1 and blocks $B_2 \cap B_1, B_3 \cap B_1, \ldots B_v \cap B_1$. The residual design with respect to B_1 has elements (varieties, points) $V - B_1$ and blocks $B_2 - B_1, B_3 - B_1, \ldots, B_v - B_1$. The incidence matrix A is a b by v matrix with rows indexed by the blocks and columns indexed by the elements with the B_i, v_j entry equal to 1 if v_j is an element of B_i and equal to 0 otherwise.

(a) Prove that in a BIBD $bk = vr$ and $r(k - 1) = \lambda(v - 1)$.
(b) Prove Fisher's inequality: In a BIBD $v \leq b$.
(c) If A is the incidence matrix of a symmetric design with block B_1 corresponding to the first row, show that the incidence matrix of the derived design is the columns of A with a 1 in the first row (with the first row deleted) and the incidence matrix of the residual design is the columns of A with a 0 in the first row (with the first row deleted).
(d) For a symmetric BIBD with parameters $b = v, r = k, \lambda$ determine the parameters b', v', r', k', λ' and $b", v", r", k", \lambda"$ for the residual and derived designs. (You may use part (c).)